Stokes and Navier-Stokes Equations with Navier Boundary Condition and some Limiting Cases

Chérif AMROUCHE

Laboratoire de Mathématiques et de leurs Applications
CNRS UMR 5142
Université de Pau et des Pays de l'Adour

Joint work with
P. ACEVEDO (Quito), C. CONCA (Santiago, Chile), A. GHOSH (Pau)

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Outline

I. Introduction and motivation

First we consider in a bounded domain Ω in $\mathbb{R}^3$ with boundary Γ, possibly not connected, of class $\mathcal{C}^{1,1}$, the following Stokes equations

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \text{div } \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega$$

where the unknowns $\boldsymbol{u}$ and π stand respectively for the velocity field and the pressure of the fluid occupying a domain Ω. Given data is the external force $\boldsymbol{f}$. To study the Stokes equations it is necessary to add some suitable boundary conditions

- Concerning these equations, the first thought goes to the classical no-slip Dirichlet boundary condition which is not always appropriate. For example it shows the absence of collisions of rigid bodies immersed in a linearly viscous fluid.
- In some applications, in particular in the electromagnetism problems, it is possible to find problems where it is necessary to consider other boundary conditions (BC). These BC are also used to simulate flows near rough walls, such as in aerodynamics, in weather forecasts and in hemodynamics, as well as perforated walls. BC involving the pressure, such as in cases of pipes, hydraulic gears using pomps, containers, etc ...

An alternative to the no-slip BC was suggested by H. Navier in 1823. Navier proposed a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangent to the surface should be proportional to the rate of strain at the surface

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad 2\left[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}\right]_{\tau} + \alpha \boldsymbol{u}_{\tau} = \boldsymbol{h} \quad \text{on } \Gamma$$

where $\mathbb{D}\boldsymbol{u} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ denotes the deformation tensor associated to the velocity field $\boldsymbol{u}$ and α is the friction coefficient which is a scalar function.

Observe that if α tends to infinity, we get formally

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on} \qquad \Gamma.$$

The Navier boundary conditions are often used to simulate flows near rough walls as well as perforated walls.

Such slip boundary conditions are used in the Large Eddy Simulations (LES) of turbulent flows.

Our aim is to study the system

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \text{div } \mathbb{F}, \quad \text{div } \boldsymbol{u} = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{h} & \text{on } \Gamma \end{cases} \tag{S}$$

or the non-linear system

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \text{div } \mathbb{F}, \quad \text{div } \boldsymbol{u} = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{h} & \text{on } \Gamma \end{cases} \tag{NS}$$

considering α regular or not regular. Here $\mathbb{F}$ is a 3×3 matrix.

We first briefly review some existing or related works.

Literature

Stationary problem :

- Solonnikov-Scadilov, 1973, $\alpha = 0$ Hilbert case
- B.Da Veiga, 2004, $\alpha > 0$ constant, Hilbert case
- Berselli, 2010, $\alpha = 0$, flat domain in $\mathbb{R}^3$
- Amrouche-Rejaiba, 2014, $\alpha = 0$
- Verfurth, 1987.

Non-stationary problem :

- Mikelić et al, 1998, 2D, $\alpha \in C^2(\Gamma)$
- Kelliher, 2006, 2D, $\alpha \in L^\infty(\Gamma)$
- B.Da Veiga, 2007, 3D, $\alpha > 0$ constant
- Iftimie-Sueur, 2011, 3D, $\alpha \in C^2(\Gamma)$

II. Basic properties and useful inequalities

To study the problem, we consider the following assumptions on α:

$$\alpha \in L^{t(p)}(\Gamma)$$

with

$$t(p) = \begin{cases} \frac{2}{3}p' + \varepsilon & \text{if} \quad 1 < p < \frac{3}{2} \\ 2 + \varepsilon & \text{if} \quad \frac{3}{2} \leq p \leq 3, p \neq 2 \\ 2 & \text{if} \quad p = 2 \\ \frac{2}{3}\, p + \varepsilon & \text{if} \quad p > 3 \end{cases}$$

where $\varepsilon > 0$ is an arbitrary small number and $\alpha \geq 0$.

Note that the kernel of the system (S) corresponding to $\alpha = 0$ is:

$$\begin{aligned} \mathcal{T}(\Omega) &= \left\{ \boldsymbol{v} \in \mathbf{W}^{1,p}(\Omega) : \mathbb{D}\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \right\} \\ &= \begin{cases} \{\mathbf{0}\} & \text{if } \Omega \text{ is not axisymmetric} \\ \text{span}\{\boldsymbol{b} \times \boldsymbol{x}\} & \text{if } \Omega \text{ is axisymmetric with axis } \boldsymbol{b} \end{cases} \end{aligned}$$

But the kernel of the system (S) for $\alpha \neq 0$ is: $\mathcal{I}(\Omega) = \{\mathbf{0}\}$.

Let us first discuss some Korn-type inequalities which will be used to prove the equivalence of norms and the existence of solution.

Proposition

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, the following equivalence of norms hold:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \qquad \textit{if } \Omega \textit{ is not axisymmetric} \,,$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \simeq \|\mathbb{D}\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{u}_\tau\|_{\mathbf{L}^2(\Gamma_0)} \qquad \textit{if } \Omega \textit{ is axisymmetric} \,.$$

where Γ_0 is a part of Γ.

We also deduce the following inequalities:

Proposition

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. If Ω is axisymmetric with respect to a constant vector $\boldsymbol{b} \in \mathbb{R}^3$ and $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b} \times \boldsymbol{x}$ for $\boldsymbol{x} \in \Omega$, then we have the following inequalities: for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ,

$$\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \left(\int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{\beta} \right)^2 \right]$$

and

$$\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \left[\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \left(\int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{\beta} \right)^2 \right].$$

These results can be proved by method of contradiction.

Now consider

$$\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega),\ \mathbb{F} \in \mathbb{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$$

s.t. $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ where

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if} \quad p > \frac{3}{2} \\ 1 & \text{if} \quad 1 < p \leq \frac{3}{2}. \end{cases}$$

Note that $\forall p \in (1, \infty)$, $r(p) < p$. We call $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ is a weak solution of the Stokes problem (S) iff for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)$,

$$2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma.$$

Also note that the boundary term in the left hand side is actually a well-defined integral with our regularity assumption on α.

It is easy to see from the above weak formulation that if $\alpha = 0$ and Ω is axisymmetric,

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

is a necessary condition for the existence of a solution.

III. L^2-Theory

The first theorem gives us the existence, uniqueness and estimates of the solution of (S).

Theorem

Let

$$\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega), \mathbb{F} \in \mathbb{L}^2(\Omega),\ \boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\Gamma) \text{ and } \alpha \in L^2(\Gamma).$$

Then the Stokes problem (S) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$ *with the following estimates:*

i) *if* Ω *is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \right)$$

ii) *if* Ω *is axisymmetric and*

- $\alpha \geq \alpha_* > 0$ *on* $\Gamma_0 \subset \Gamma$, *then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega, \alpha_*) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \right)$$

- ***f***, ***h*** *satisfy the condition:*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$$

then, the solution $\boldsymbol{u}$ *satisfies* $\int_\Gamma \alpha \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \int_\Gamma \alpha |\boldsymbol{u}_\tau|^2 \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \right)^2.$$

In particular, if α *is a non-zero constant, then* $\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{\beta} = 0$ *and*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \right).$$

Proof. The existence and uniqueness follows from the Lax-Milgram Lemma (where the coercivity of the bilinear form is obvious) and also the estimate. But note that the continuity constant we get from Lax-Milgram Lemma depends on α. So we prove separately the different estimates, independent of α. For that we use the previously stated Korn-type inequalities and equivalence of norms.

Next we prove the existence of strong solution in the Hilbert case and the corresponding estimate independent of α. Later we will prove the existence of strong solution for general α, not necessarily a constant.

Theorem

Let $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\Gamma)$ and α be a constant. Then the solution of (S) with $\mathbb{F} = 0$ belongs to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ satisfying the estimates,

(i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \right).$$

(ii) *if Ω is axisymmetricc, then*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha\}} \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \right).$$

If moreover $\boldsymbol{f}, \boldsymbol{h}$ satisfy the condition: $\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta} \rangle_\Gamma = 0$, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{1/2}(\Gamma)} \right).$$

Proof.

- If α is a constant, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{1/2}(\Gamma)$, we have the existence of a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ which implies $\alpha \boldsymbol{u}_\tau \in \boldsymbol{H}^{1/2}(\Gamma)$. Then from the regularity result in Amrouche-Rejaiba, JDE (2014), we obtain $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.
- But concerning the estimate, with this method, we do not get bound on $\boldsymbol{u}$, independent of α. Thus we consider the more fundamental but long method of difference quotient which essentially uses $\boldsymbol{H}^1$-estimate and thus we obtain uniform bounds with respect to α.
- Note that to show $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{H}^2_{loc}(\Omega) \times H^1_{loc}(\Omega)$ with the corresponding estimate is classical, since to obtain the interior regularity does not depend on the boundary condition.

1. Interior regularity. Showing that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{H}^2_{loc}(\Omega) \times H^1_{loc}(\Omega)$, is very classical, with the help of $\boldsymbol{H}^1$-estimate, since the method does not depend on the boundary condition.

2. Boundary regularity.

The solution $(\boldsymbol{u}, \pi)$ satisfies the variational formulation, for all $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\tau(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \qquad (0.1)$$

Case 1 : Ω is a half-ball i.e. $\Omega = B(0,1) \cap \mathbb{R}^3_+$.
Set $V := B(0, \frac{1}{2}) \cap \mathbb{R}^3_+$. Then choose a cut-off function $\zeta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{cases} \zeta \equiv 1 \text{ on } B(0, \frac{1}{2}),\ \zeta \equiv 0 \textbf{ on } \mathbb{R}^3 \setminus B(0,1), \\ 0 \leq \zeta \leq 1. \end{cases}$$

So $\zeta \equiv 1$ on V and vanishes on the curved part of Γ.

- **tangential regularity of $\boldsymbol{u}$:**

Let $h > 0$ be small and $\boldsymbol{\varphi} = -D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}), k = 1, 2$. Clearly, $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\tau(\Omega)$. Therefore, we can substitute $\boldsymbol{\varphi}$ into the identity (0.1) and obtain,

$$2\int_\Omega \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + 2\int_\Omega D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta\nabla\zeta D_k^h \boldsymbol{u} + \int_\Gamma \alpha\zeta^2 |D_k^h \boldsymbol{u}_\tau|^2 - \int_\Omega \pi \operatorname{div}(-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})) = \int_\Omega \boldsymbol{f} \cdot (-D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u})). \tag{0.2}$$

Now we estimate the different terms. For the second term in the left hand side, using Cauchy's inequality with ε, we get

$$\left|\int_\Omega D_k^h \mathbb{D}\boldsymbol{u} : 2\zeta\nabla\zeta D_k^h \boldsymbol{u}\right| \leq C\int_\Omega 2\zeta |D_k^h \mathbb{D}\boldsymbol{u}||D_k^h \boldsymbol{u}| \leq C\left[\varepsilon\int_\Omega \zeta^2 |D_k^h \mathbb{D}\boldsymbol{u}|^2 + \frac{1}{\varepsilon}\int_\Omega |D_k^h \boldsymbol{u}|^2\right].$$

Similarly we can estimate the fourth term in the left hand side and the term in the right hand side which gives us altogether

$$2\int_\Omega \zeta^2|D_k^h \mathbb{D}\boldsymbol{u}|^2 + \int_\Gamma \alpha\ \zeta^2|D_k^h \boldsymbol{u}_\tau|^2$$

$$\leq \varepsilon \int_\Omega \zeta^2|\nabla D_k^h \boldsymbol{u}|^2 + \frac{C_1}{\varepsilon}\left(\int_\Omega |\boldsymbol{f}|^2 + \int_\Omega |\pi|^2\right) + C_2 \int_\Omega |D_k^h \boldsymbol{u}|^2.$$

From here, we deduce

$$\|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \varepsilon \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \frac{C_1}{\varepsilon}\left(\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\pi\|^2_{\mathbf{L}^2(\Omega)}\right) + C_2\|D_k^h \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}$$

Choosing ε small and using the estimates for $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$, we obtain,

$$\|D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(V)} \leq \|\zeta D_k^h \boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega)\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)}$$

which means that $\partial^2 \boldsymbol{u}/\partial x_i \partial x_j$ belongs to $\mathbf{L}^2(V)$ for all $i, j = 1, 2, 3$ except for $i = j = 3$, with the corresponding estimate.

- **tangential regularity of π :** From the Stokes equation, for $i = 1, 2$, we get,

$$\frac{\partial}{\partial x_i}(\nabla \pi) = \frac{\partial}{\partial x_i}(\boldsymbol{f} + \Delta \boldsymbol{u}) = \frac{\partial \boldsymbol{f}}{\partial x_i} + \operatorname{div}(\nabla \frac{\partial \boldsymbol{u}}{\partial x_i}).$$

Since there is no term of the form $\partial^2 \boldsymbol{u}/\partial x_3^2$, by preceding arguments, we obtain

$$\nabla \frac{\partial \pi}{\partial x_i} = \frac{\partial}{\partial x_i}(\nabla \pi) \in \boldsymbol{H}^{-1}(V).$$

Furthermore, as we already know that $\frac{\partial \pi}{\partial x_i} \in H^{-1}(V)$, hence from the Nečas inequality, $\frac{\partial \pi}{\partial x_i} \in L^2(V)$ and satisfies the usual estimate.

- **normal regularity :** For the complete regularity of the solution, it remains to study the derivatives of $\boldsymbol{u}$ and π in the direction $\mathbf{e}_3$. Differentiating the divergence equation with respect to x_3 gives,

$$\frac{\partial^2 u_3}{\partial x_3^2} = -\sum_{i=1}^{2} \frac{\partial^2 u_i}{\partial x_i \partial x_3} \in L^2(V).$$

Next, from the 3rd component of the Stokes equation, we can write,

$$\frac{\partial \pi}{\partial x_3} = f_3 + \Delta u_3 \in L^2(V)$$

which proves that $\pi \in H^1(V)$. Finally, for $i = 1, 2$, we can write the ith equation of the system in the form,

$$\frac{\partial^2 u_i}{\partial x_3^2} = -\sum_{j=1}^{2} \frac{\partial^2 u_i}{\partial x_j^2} - f_i + \frac{\partial \pi}{\partial x_i} \in L^2(V).$$

Thus, $u_i \in H^2(V)$. So, apart from the regularity of $\boldsymbol{u}$ and π, we have proved the existence of $C = C(\Omega) > 0$ independent of α such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} + \|\pi\|_{H^1(V)} \leq C\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}.$$

Case 2 : Now we drop the assumption that Ω is a half ball and consider the general case. Since Γ is $\mathcal{C}^{2,1}$, for any $x_0 \in \Gamma$, we can assume, upon relabelling the coordinate axes,

$$\Omega \cap (B(x_0, r)) = \{x \in (B(x_0, r)) : x_3 > H(x')\}$$

for some $r > 0$ and $H : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{2,1}$.

Let us now introduce the change of variable,

$$y = (x_1, x_2, x_3 - H(x')) := \phi(x)$$

i.e.

$$x = (y_1, y_2, y_3 + H(y')) := \phi^{-1}(y).$$

Choose $s > 0$ small so that the half ball $\Omega' := B(0, s) \cap \mathbb{R}^3_+$ lies in $\phi(\Omega \cap (B(x_0, r)))$. Set also $V' := B(0, s/2) \cap \mathbb{R}^3_+$ and $\boldsymbol{u}'(y) = \boldsymbol{u}(\phi^{-1}(y))$ for $y \in \Omega'$. It is easy to see that

$$\boldsymbol{u}' \in \boldsymbol{H}^1(\Omega')$$

and

$$\boldsymbol{u}' \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega' \cap \partial\mathbb{R}^3_+$$

but

$$\text{div } \boldsymbol{u} = \text{div } \boldsymbol{u}' - \sum_{j=1}^{2} \frac{\partial H}{\partial y_j} \frac{\partial u'_j}{\partial y_3}.$$

Now transforming our problem to the new coordinates, under this change of variable, (0.1) becomes,

$$
\begin{aligned}
&2\int_{\Omega'} \mathbb{D}\boldsymbol{u}' : \mathbb{D}\boldsymbol{\varphi}' \,\mathrm{d}y + \int_{\Gamma'} \alpha \boldsymbol{u}'_\tau \cdot \boldsymbol{\varphi}'_\tau \,\mathrm{d}s - \int_{\Omega'} \pi' \,\mathrm{div}\boldsymbol{\varphi}' \,\mathrm{d}y \\
&= \int_{\Omega'} \boldsymbol{f}' \cdot \boldsymbol{\varphi}' \,\mathrm{d}y - \int_{\Omega'} \pi' \frac{\partial H}{\partial y_j}\frac{\partial u'_j}{\partial y_3} \,\mathrm{d}y + \int_{\Omega'} \partial H \;\nabla \boldsymbol{u}' \nabla \boldsymbol{\varphi}' \,\mathrm{d}y.
\end{aligned} \tag{0.3}
$$

We choose the test function $\boldsymbol{\varphi}' = -D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}')$ as before and estimate the extra terms. For the second term in the right hand side, it is easy to see,

$$
|\int_{\Omega'} \pi' \frac{\partial H}{\partial y_j}\frac{\partial u'_j}{\partial y_3} \,\mathrm{d}y| \leq C\left(\|\pi'\|^2_{L^2(\Omega')} + \|\nabla \boldsymbol{u}'\|^2_{\mathbf{L}^2(\Omega')} \right).
$$

And for the last term in the right hand side, we get

$$
\begin{aligned}
&|\int_{\Omega'} \partial H \;\nabla \boldsymbol{u}' \nabla (D_k^{-h}(\zeta^2 D_k^h \boldsymbol{u}')) \,\mathrm{d}y| \\
\leq & C\left(\int_{\Omega'} |\nabla \boldsymbol{u}'|^2 \,\mathrm{d}y + \varepsilon \int_{\Omega'} \zeta^2 |\nabla D_k^h \boldsymbol{u}'|^2 \,\mathrm{d}y + \frac{1}{\varepsilon}\int_{\Omega'} |D_k^h \boldsymbol{u}'|^2 \,\mathrm{d}y \right. \\
&\left. + \int_{\Omega'} \zeta^2 |D_k^h \nabla \boldsymbol{u}'|^2 |\partial H| \,\mathrm{d}y \right).
\end{aligned}
$$

Hence, accumulating all these inequalities, we obtain from (0.3),

$$\|\zeta D_k^h \boldsymbol{u}'\|^2_{\boldsymbol{H}^1(\Omega')} \leq C\left(\|\nabla \boldsymbol{u}'\|^2_{\mathbf{L}^2(\Omega')} + \|\pi'\|^2_{L^2(\Omega')} + \|\boldsymbol{f}'\|^2_{\mathbf{L}^2(\Omega')}\right.$$
$$\left. + \varepsilon\|\zeta D_k^h \boldsymbol{u}'\|^2_{\boldsymbol{H}^1(\Omega')} + \int_{\Omega'} \zeta^2 |\partial H||\nabla D_k^h \boldsymbol{u}'|^2 \,\mathrm{d}y\right).$$

But $|\partial H|$ is small for sufficiently small $s > 0$, since

$$\frac{\partial H}{\partial y_1}(0,0) = 0 = \frac{\partial H}{\partial y_2}(0,0)$$

which gives,

$$\|D_k^h \boldsymbol{u}'\|^2_{\boldsymbol{H}^1(V')} \leq \|\zeta D_k^h \boldsymbol{u}'\|^2_{\boldsymbol{H}^1(\Omega')} \leq C\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)}. \tag{0.4}$$

Then, proceeding as in the case of the half ball, we can deduce,

$$\boldsymbol{u}' \in \boldsymbol{H}^2(V') \quad \text{and} \quad \|\boldsymbol{u}'\|_{\boldsymbol{H}^2(V')} \leq C(\Omega)\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}.$$

Consequently,

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(V)} \leq C(\Omega)\|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}.$$

Since, Γ is compact, we can cover Γ with finitely many sets $\{V_i\}$ as above. Summing the resulting estimates, along with the interior estimate, we get $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ with the desired inequality.

IV. L^p-Theory

We begin with recalling some results, required for our study.

Theorem (inf-sup condition in Banach spaces)

Let X and M be two reflexive Banach spaces and X' and M' be their dual spaces. Let a be the continuous bilinear form defined on $X \times M$, $A \in \mathcal{L}(X; M')$ and $A' \in \mathcal{L}(M; X')$ be the operators defined by

$$\forall v \in X, \quad \forall w \in M, \quad a(v, w) = \langle Av, w \rangle = \langle v, A'w \rangle$$

and $V = Ker\ A$. Then the following statements are equivalent:
(i) There exists $C > 0$ such that

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v, w)}{\|v\|_X \ \|w\|_M} \geq C. \tag{0.5}$$

(ii) The operator $A : X/V \mapsto M'$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of A^{-1}.
(iii) The operator $A' : M \mapsto X' \perp V$ is an isomorphism and $\frac{1}{C}$ is the continuity constant of $(A')^{-1}$.

Also, recall the following inf-sup condition derived in Amrouche-Seloula (2013, M3AS):

Lemma

There exists a constant $\gamma > 0$ *such that*

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq \mathbf{0}}} \sup_{\substack{\boldsymbol{\xi}\in \mathbf{W}^{1,p}_{\sigma,\tau}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{V}^{p'}(\Omega)}} \geq \gamma \qquad (0.6)$$

where

$$\boldsymbol{V}^{p'}(\Omega) := \left\{\boldsymbol{v}\in \mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega);\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0 \quad \forall\ 1\leq j\leq J\right\}.$$

Next we consider the bilinear form: $\forall \boldsymbol{u}\in \mathbf{W}^{1,p}_{\sigma,\tau}(\Omega), \boldsymbol{\varphi}\in \mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)$,

$$a(\boldsymbol{u},\boldsymbol{\varphi}) = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha\boldsymbol{u}_\tau\cdot\boldsymbol{\varphi}_\tau$$

and prove a general result.

Theorem

Let $1 < p < \infty$, and

$$\alpha \in L^{t(p)}(\Gamma) \quad \text{and} \quad \boldsymbol{\ell} \in [\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)]'.$$

Then the problem:

$$\textit{find } \boldsymbol{u} \in \mathbf{W}^{1,p}_{\sigma,\tau}(\Omega) \textit{ s.t. for any } \boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega), \; a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle \quad (0.7)$$

has a unique solution.

Remark. Note that for any $\boldsymbol{\ell} \in [\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)]'$, we can not interpret the solution $\boldsymbol{u}$ of (0.7) as a solution of the Stokes problem (S) in general since $[\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)]' \not\subset \mathcal{D}'(\Omega)$.

Proof. First consider $p \geq 2$. Since $[\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)]' \hookrightarrow [\mathbf{H}^1_{\sigma,\tau}(\Omega)]'$, by Lax-Milgram lemma there exists a unique $\boldsymbol{u} \in \mathbf{H}^1_{\sigma,\tau}(\Omega)$ satisfying:

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^1_{\sigma,\tau}(\Omega), \quad a(\boldsymbol{u}, \boldsymbol{\varphi}) = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\mathbf{H}^1_{\sigma,\tau}(\Omega)]' \times \mathbf{H}^1_{\sigma,\tau}(\Omega)}. \qquad (0.8)$$

Now we will prove that $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$. Since the inf-sup condition (0.5) is known for the bilinear form

$$b(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\ \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi}$$

with adapted spaces X and M and we have the relation

$$2\,[(\mathbb{D}\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} - 2\ \boldsymbol{\Lambda}\boldsymbol{u} \quad \text{on}\,\Gamma$$

where $\boldsymbol{\Lambda}$ is an operator of order 0 defined by

$$\boldsymbol{\Lambda}\boldsymbol{u} = \sum_{k=1}^{2} \Big(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\Big)\boldsymbol{\tau}_k,$$

we will use another formulation of Problem (0.8).

Using the density result,

$$\{\boldsymbol{v} \in \boldsymbol{H}^2(\Omega), \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\} \text{ is dense in } \mathbf{H}^1_{\sigma,\tau}(\Omega),$$

we can write (0.8) as, for all $\boldsymbol{\varphi} \in \mathbf{H}^1_{\sigma,\tau}(\Omega)$,

$$\int_\Omega \mathbf{curl}\ \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle_{[\mathbf{H}^1_{\sigma,\tau}(\Omega)]' \times \mathbf{H}^1_{\sigma,\tau}(\Omega)} - \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi}. \tag{0.9}$$

Now we are in position to improve the regularity of $\boldsymbol{u}$ and for that we use bootstrap argument.

case (i): $2 < p \leq 3$.

Step 1. Since $\boldsymbol{u}_\tau \in \mathbf{L}^4(\Gamma)$ and $\alpha \in L^{2+\varepsilon}(\Gamma)$, we have $\alpha \boldsymbol{u}_\tau \in \mathbf{L}^{q_1}(\Gamma)$ where $\frac{1}{q_1} = \frac{1}{4} + \frac{1}{2+\varepsilon}$. But, $\mathbf{L}^{q_1}(\Gamma) \hookrightarrow \mathbf{W}^{-\frac{1}{p_1}, p_1}(\Gamma)$ with $p_1 = \frac{3}{2} q_1 > 2$ i.e. $\frac{1}{p_1} = \frac{2}{3}\left(\frac{1}{4} + \frac{1}{2+\varepsilon}\right)$. Therefore, as $\mathbf{W}^{\frac{1}{p_1}, p_1'}(\Gamma) \hookrightarrow \mathbf{L}^{q_1'}(\Gamma)$ with $\frac{4}{3} < q_1' < 4$ and $\boldsymbol{\Lambda} \boldsymbol{u} \in \mathbf{L}^4(\Gamma)$, the mapping

$$\langle \mathbf{L}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle - \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi} \quad \text{for} \quad \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega)$$

defines an element of the dual space of $\boldsymbol{V}^{s_1'}(\Omega)$ with $s_1 = \min\{p_1, p\}$.

Now from the Inf-Sup condition (0.6), $\exists$ a unique $\boldsymbol{v} \in \mathbf{W}^{1,s_1}_{\sigma,\tau}(\Omega)$ s.t.

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^{s_1'}(\Omega), \quad \int_\Omega \mathbf{curl}\ \boldsymbol{v} \cdot \mathbf{curl}\ \boldsymbol{\varphi} = \langle \mathbf{L}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{V}^{s_1'}(\Omega)]' \times \boldsymbol{V}^{s_1'}(\Omega)}. \tag{0.10}$$

We will show that $\mathbf{curl}\ \boldsymbol{v} = \mathbf{curl}\ \boldsymbol{u}$. For that first we extend (0.10) to any test function $\boldsymbol{\varphi} \in \mathbf{W}^{1,s_1'}_{\sigma,\tau}(\Omega)$ and since $\mathbf{H}^1_{\sigma,\tau}(\Omega) \hookrightarrow \mathbf{W}^{1,s_1'}_{\sigma,\tau}(\Omega)$, we deduce from (0.9) that

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^1_{\sigma,\tau}(\Omega), \quad \int_\Omega \mathbf{curl}\ \boldsymbol{v} \cdot \mathbf{curl}\ \boldsymbol{\varphi} = \int_\Omega \mathbf{curl}\ \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi}$$

which gives,

$$\mathbf{curl}\ \boldsymbol{u} = \mathbf{curl}\ \boldsymbol{v} \quad \text{in} \ \ \Omega\ .$$

Therefore, as $\boldsymbol{u} \in \mathbf{L}^6(\Omega) \hookrightarrow \mathbf{L}^{s_1}(\Omega), \mathbf{curl}\ \boldsymbol{u} \in \mathbf{L}^{s_1}(\Omega), \text{div}\ \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ; we get $\boldsymbol{u} \in \mathbf{W}^{1,s_1}(\Omega)$. If $s_1 \geq p$, the proof is complete. Otherwise, $s_1 = p_1$ and we proceed to the next step.

Step 2. Now, $\boldsymbol{u} \in \mathbf{W}^{1,p_1}(\Omega)$ implies the mapping

$$\langle \mathbf{L}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle - \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi} \quad \text{for} \ \ \boldsymbol{\varphi} \in \boldsymbol{V}^{s_2'}(\Omega)$$

defines an element of the dual space of $\boldsymbol{v}^{s_2'}(\Omega)$ with $s_2 = \min\{p_2, p\}$ where $\frac{1}{p_2} = \frac{2}{3}\left(\frac{2}{2+\varepsilon} - \frac{1}{2} + \frac{1}{4}\right)$. Therefore, as in the previous step, $\exists$ a unique $\boldsymbol{v} \in \mathbf{W}^{1,s_2}_{\sigma,\tau}(\Omega)$ s.t.

$$\forall\ \boldsymbol{\varphi} \in \mathbf{W}^{1,s_2'}_{\sigma,\tau}(\Omega), \qquad \int_\Omega \mathbf{curl}\ \boldsymbol{v} \cdot \mathbf{curl}\ \boldsymbol{\varphi} = \langle \mathbf{L}, \boldsymbol{\varphi} \rangle$$

which implies again

$$\mathbf{curl}\ \boldsymbol{u} = \mathbf{curl}\ \boldsymbol{v} \quad \text{in } \Omega.$$

Therefore, we get, $\boldsymbol{u} \in \mathbf{L}^{p_1^*}(\Omega) \hookrightarrow \mathbf{L}^{s_2}(\Omega)$, $\mathbf{curl}\ \boldsymbol{u} \in \mathbf{L}^{s_2}(\Omega)$, $\text{div}\ \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ; which implies $\boldsymbol{u} \in \mathbf{W}^{1,s_2}(\Omega)$. If $s_2 \geq p$, we are done. Otherwise, $s_2 = p_2$ and we proceed next.

Step (k+1). Proceeding similarly, we get $\boldsymbol{u} \in \mathbf{W}^{1,p_{k+1}}_{\sigma,\tau}(\Omega)$ with $\frac{1}{p_{k+1}} = \frac{2}{3}\left(\frac{k+1}{2+\varepsilon} - \frac{k}{2} + \frac{1}{4}\right)$ (where in each step, we assumed $p_k < 3$) which also satisfies, for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,p'_{k+1}}_{\sigma,\tau}(\Omega)$,

$$\int_\Omega \mathbf{curl}\ \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi} = \langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle - \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau + 2 \int_\Gamma \boldsymbol{\Lambda} \boldsymbol{u} \cdot \boldsymbol{\varphi}.$$

Now choose $k = [\frac{1}{\varepsilon} - \frac{1}{2}] + 1$ such that $p_{k+1} \geq 3 \geq p$ (where $[a]$ stands for the greatest integer less than or equal to a). Hence $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$.

Case (II) : $p > 3$.

From the previous case, we get that $\boldsymbol{u} \in \mathbf{W}^{1,3}(\Omega)$ which implies $\boldsymbol{u}_\tau \in \mathbf{L}^s(\Gamma)$ for all $1 < s < \infty$. Then by similar argument (and in one iteration) we can deduce $\boldsymbol{u} \in \mathbf{W}^{1,p}_{\sigma,\tau}(\Omega)$ which solves the problem (0.7).

- p<2.

Consider the operator $A \in \mathcal{L}(\mathbf{W}^{1,p}_{\sigma,\tau}(\Omega), (\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega))')$, associated to the bilinear form a, defined as, $\langle A\xi, \varphi\rangle = a(\xi, \varphi)$. As described above, for $p \geq 2$, the operator A is an isomorphism from $\mathbf{W}^{1,p}_{\sigma,\tau}(\Omega)$ to $(\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega))'$. Then the adjoint operator, which is equal to A is an isomorphism from $\mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega)$ to $(\mathbf{W}^{1,p}_{\sigma,\tau}(\Omega))'$ for $p' \leq 2$. This means that the operator A is an isomorphism for $p \leq 2$ also, which ends the proof.

From the equivalence of the general inf-sup theorem (stated before), we obtain the following important inf-sup condition:

inf-sup condition: for any $p \in (1, \infty)$ and $\alpha \in L^{t(p)}(\Gamma)$, there exists $\gamma = \gamma(\Omega, p, \alpha) > 0$ such that

$$\inf_{\substack{\boldsymbol{\varphi} \in \mathbf{W}^{1,p'}_{\sigma,\tau}(\Omega \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \mathbf{W}^{1,p}_{\sigma,\tau}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{2 \int_\Omega \mathbb{D}\boldsymbol{\xi} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{\xi}_\tau \cdot \boldsymbol{\varphi}_\tau}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \gamma. \qquad (0.11)$$

In particular, choosing $\langle \boldsymbol{\ell}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma$ in problem (0.7), we get the following existence result:

Theorem

Let $p \in (1, \infty)$ and

$$\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma), \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

Then the Stokes problem (S) *has a unique solution* $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\alpha, \Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

Question: Is it possible to obtain γ in the above inf-sup condition independent of α, or equivalently the constant C in the above estimate independent of α?

We will come to it later!

Theorem (Strong solution)

Let $p \in (1, \infty)$ and $\boldsymbol{f} \in \mathbf{L}^p(\Omega), \boldsymbol{h} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma)$ and

$$\alpha \in \begin{cases} W^{1-\frac{1}{\frac{3}{2}+\varepsilon},\frac{3}{2}+\varepsilon}(\Gamma) & \text{if} \quad 1 < p \leq \frac{3}{2} \\ W^{1-\frac{1}{p},p}(\Gamma) & \text{if} \quad p > \frac{3}{2}. \end{cases}$$

Then the unique solution $(\boldsymbol{u}, \pi)$ of the Stokes problem (S) with $\mathbb{F} = 0$ belongs to $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Question: Is it possible to obtain γ in the above inf-sup condition independent of α, or equivalently the constant C in the above estimate independent of α?

We will come to it later!

Theorem (Strong solution)

Let $p \in (1, \infty)$ *and* $\boldsymbol{f} \in \mathbf{L}^p(\Omega), \boldsymbol{h} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma)$ *and*

$$\alpha \in \begin{cases} W^{1-\frac{1}{\frac{3}{2}+\varepsilon},\frac{3}{2}+\varepsilon}(\Gamma) & \text{if} \quad 1 < p \leq \frac{3}{2} \\ W^{1-\frac{1}{p},p}(\Gamma) & \text{if} \quad p > \frac{3}{2}. \end{cases}$$

Then the unique solution $(\boldsymbol{u}, \pi)$ *of the Stokes problem* (S) *with* $\mathbb{F} = 0$ *belongs to* $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

Estimates

Now concerning the α independent estimate for general solution, we first discuss the estimate for $p > 2$ with $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{h} = \mathbf{0}$, similar to the Hilbert case.

Theorem

Let $p > 2$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then the solution $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ of (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$ satisfies the following estimates:

i) *if Ω is not axisymmetric, then*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_p(\Omega) \ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}$$

ii) *if Ω is axisymmetric and $\alpha \geq \alpha_* > 0$, then*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_p(\Omega, \alpha_*) \ \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}.$$

The proof of the above theorem uses the weak Reverse Hölder inequality and is similar to the result done for the the Stokes problem with Dirichlet boundary condition (cf. Giaquinta-Modica).

Also we need the following lemma (cf. Giaquinta-Modica) to deduce weak Reverse Hölder inequality, which we call "epsilon lemma":

Lemma

Let f, g, h be non-negative functions in $L^1(Q_0)$ where Q_0 is a cube in $\mathbb{R}^n$, $Q_R(x_0)$ is a cube centered at x_0 with sides $2R$ and let $\beta \in \mathbb{R}^+$. There exists δ_0 such that if for some $\delta \leq \delta_0$, the following inequality

$$\int_{Q_R(x_0)} f \leq C(\delta) \left[R^{-\beta} \int_{Q_{2R}(x_0)} g + \int_{Q_{2R}(x_0)} h \right] + \delta \int_{Q_{2R}(x_0)} f$$

holds for all $x_0 \in Q_0$ and $R < \frac{1}{2} d(x_0, \partial Q_0)$, then there exists a constant $C > 0$ such that

$$\int_{Q_R(x_0)} f \leq C \left[R^{-\beta} \int_{Q_{2R}(x_0)} g + \int_{Q_{2R}(x_0)} h \right]$$

for all $x_0 \in Q_0$ and all $R < \frac{1}{2} d(x_0, \partial Q_0)$.

Proof

Since Ω is $\mathcal{C}^{1,1}$, there exists $r_0 > 0$ s.t. for any $x_0 \in \Gamma$, there exists a coordinate system (x', x_3) which is isometric to the usual coordinate system and a $\mathcal{C}^{1,1}$ function $\psi : \mathbb{R}^2 \to \mathbb{R}$ so that

$$B(x_0, r_0) \cap \Omega = \{(x', x_3) \in B(x_0, r_0) : x_3 > \psi(x')\}$$

and

$$B(x_0, r_0) \cap \Gamma = \{(x', x_3) \in B(x_0, r_0) : x_3 = \psi(x')\} .$$

Now consider any ball $B(x_0, r)$ with the property that $0 < r < \frac{r_0}{8}$ and either $B(x_0, 2r) \subset \Omega$ or $x_0 \in \Gamma$. Here onwards, we denote $aB := B(x_0, ar)$ for $a > 0$ and $\fint_\omega f = \frac{1}{|\omega|} \int_\omega f$.

Multiplying (S) with $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$ by a test function $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$ with $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ, we obtain,

$$2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau - \int_\Omega \pi \text{ div } \boldsymbol{\varphi} = - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi}. \qquad (0.12)$$

Now we establish some weak Reverse Hölder inequality.

a) Pressure estimate:

Let $\pi_0 = \fint_{2B\cap\Omega} \pi$. Since $\pi - \pi_0 \in L^2(2B\cap\Omega)$, there exists a unique $\psi \in \boldsymbol{H}^1(2B\cap\Omega)$ such that

$$\begin{cases} \text{div } \psi = \pi - \pi_0 & \text{in } 2B\cap\Omega \\ \psi = \boldsymbol{0} & \text{on } \partial(2B\cap\Omega) \end{cases}$$

satisfying

$$\|\psi\|_{\boldsymbol{H}^1(2B\cap\Omega)} \leq C(\Omega)\ \|\pi - \pi_0\|_{L^2(2B\cap\Omega)}.$$

Now putting ψ as a test function in the above variational formulation (0.12) (extending by 0 outside $2B\cap\Omega$, we can consider $\psi \in \boldsymbol{H}_0^1(\Omega)$) and replacing π by $\pi - \pi_0$, we obtain

$$2\int_{2B\cap\Omega} \mathbb{D}\boldsymbol{u} : \mathbb{D}\psi - \int_{2B\cap\Omega} (\pi - \pi_0) \text{ div } \psi = -\int_{2B\cap\Omega} \mathbb{F} : \nabla\psi$$

which implies

$$\|\pi - \pi_0\|_{L^2(2B\cap\Omega)} \leq C(\Omega) \left(\|\mathbb{D}\boldsymbol{u}\|_{\mathbb{L}^2(2B\cap\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(2B\cap\Omega)}\right). \qquad (0.13)$$

b) Caccioppoli inequality:

Consider a cut-off unction $\eta \in C_c^\infty(2B)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B \quad \text{and} \quad |\nabla \eta| \leq \frac{C}{r} \text{ in } 2B.$$

and putting $\boldsymbol{\varphi} = \eta^2 \boldsymbol{u}$ as a test function in the variational formulation (0.12), we get,

$$\int_{B\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 + \int_{B\cap\Gamma} \alpha |\boldsymbol{u}_\tau|^2 \leq \frac{C}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \varepsilon \int_{2B\cap\Omega} |\pi - \pi_0|^2 + \varepsilon \int_{2B\cap\Omega} |\nabla \boldsymbol{u}|^2 + C \int_{2B\cap\Omega} |\mathbb{F}|^2$$

where the constant $C > 0$ is independent of α. Now plugging the pressure estimate (0.13) and using $\alpha \geq 0$ gives,

$$\int_{B\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 \leq \frac{C}{r^2} \int_{2B\cap\Omega} |\boldsymbol{u}|^2 + \varepsilon \int_{2B\cap\Omega} |\mathbb{D}\boldsymbol{u}|^2 + \varepsilon \int_{2B\cap\Omega} |\nabla \boldsymbol{u}|^2 + C \int_{2B\cap\Omega} |\mathbb{F}|^2.$$

Next adding the term $\int_{B\cap\Omega} |\boldsymbol{u}|^2$ on both sides, choosing $r \leq 1$ (as Ω is bounded, we can do so) and using Korn inequality, we obtain

$$\|\boldsymbol{u}\|^2_{\mathbf{H}^1(B\cap\Omega)} \leq C(\Omega)\left(\frac{1}{r^2}\int_{2B\cap\Omega}|\boldsymbol{u}|^2 + \int_{2B\cap\Omega}|\mathbb{F}|^2\right) + \varepsilon\|\boldsymbol{u}\|^2_{\mathbf{H}^1(2B\cap\Omega)}.$$

Here we need to use the "epsilon Lemma" to deduce

$$\int_{B\cap\Omega}|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2 \leq C(\Omega)\left(\frac{1}{r^2}\int_{2B\cap\Omega}|\boldsymbol{u}|^2 + \int_{2B\cap\Omega}|\mathbb{F}|^2\right)$$

which is the Caccioppoli inequality for Stokes problem (S).

c) Reverse Hölder inequality: Using Sobolev estimate in the above inequality, we get,

$$\int_{B\cap\Omega}|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2 \leq C(\Omega)\left(\frac{1}{r^2}\,\|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,6/5}(2B\cap\Omega)} + \int_{2B\cap\Omega}|\mathbb{F}|^2\right)$$

$$\leq C(\Omega)\left[\frac{1}{r^2}\left(\int_{2B\cap\Omega}\left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{3/5}\right)^{5/3} + \int_{2B\cap\Omega}|\mathbb{F}|^2\right].$$

Upon normalizing, this implies

$$\fint_{B\cap\Omega} |\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2 \leq C(\Omega)\left[\left(\fint_{2B\cap\Omega} \left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{3/5}\right)^{5/3} + \fint_{2B\cap\Omega} |\mathbb{F}|^2\right].$$

Now applying Gehring's result, we get improved integrability, for some $s > 2$,

$$\left(\fint_{B\cap\Omega} (|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2)^{s/2}\right)^{1/s} \leq C_s(\Omega)\left[\left(\fint_{2B\cap\Omega} |\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{1/2} + \left(\fint_{2B\cap\Omega} |\mathbb{F}|^s\right)^{1/s}\right].$$

Repeating Gehring's result finite times, we get for $p > 2$,

$$\left(\fint_{B\cap\Omega} \left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{p/2}\right)^{1/p} \leq C_p(\Omega)\left[\left(\fint_{2B\cap\Omega} |\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{1/2} + \left(\fint_{2B\cap\Omega} |\mathbb{F}|^p\right)^{1/p}\right]$$

which is the weak Reverse Hölder estimate.

Finally simplifying the last inequality yields,

$$\left(\int_{B\cap\Omega} \left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)^{p/2}\right)^{1/p} \le C_p(\Omega)\left[r^{3(p-2)}\left(\int_{2B\cap\Omega} \left(|\boldsymbol{u}|^2 + |\mathbb{D}\boldsymbol{u}|^2\right)\right)^{1/2} + \left(\int_{2B\cap\Omega} |\mathbb{F}|^p\right)^{1/p}\right].$$

Since Ω is a bounded domain, it can be covered by a finitely many balls $B(x,r)$ with $r \le 1$ and thus we can write,

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \le C_p(\Omega)\left(\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)}\right)$$

which along with the L^2-bounds give the required α independent estimates.

Next we extend the above result for $p < 2$ by using dual problem and then for $\boldsymbol{f} \neq \boldsymbol{0}$ ans $\boldsymbol{h} \neq \boldsymbol{0}$, namely:

Theorem

Let $p \in (1, \infty)$ and

$$\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^{p}(\Omega), \boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma), \alpha \in L^{t(p)}(\Gamma).$$

Then the solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^{p}_{0}(\Omega)$ of the Stokes problem (S) satisfies the following estimates, in the case either

i) Ω *is not axisymmetric*

or

ii) Ω *is axisymmetric and* $\alpha \geq \alpha_* > 0$,

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C_p(\Omega) \left(\|\boldsymbol{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} \right).$$

Thus, we can finally show that the continuity constant γ in the inf-sup condition is actually independent of α in the case either (i) Ω is not axisymmetric or (ii) Ω is axisymmetric and $\alpha \geq \alpha_* > 0$ on Γ.

V. Limiting cases

Now we discuss some limiting cases for the system

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f} + \text{div } \mathbb{F}, \quad \text{div } \boldsymbol{u}_\alpha = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{u}_\alpha \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_\alpha + \mathbb{F})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_{\alpha\tau} = \boldsymbol{h} & \text{on } \Gamma \end{cases} \tag{0.14}$$

Theorem (α tends to 0)

Let $p \in (1, \infty)$, Ω be not axisymmetric and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the solution of (0.14) where $\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $\boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$ and $\alpha \in L^{t(p)}(\Gamma)$. Then as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_0, \pi_0) \quad \textit{in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

where $(\boldsymbol{u}_0, \pi_0)$ is a solution of the following system

$$\begin{cases} -\Delta \boldsymbol{u}_0 + \nabla \pi_0 = \boldsymbol{f} + \text{div } \mathbb{F}, \quad \text{div } \boldsymbol{u} = 0 & \textit{in } \Omega \\ \boldsymbol{u}_0 \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u}_0 + \mathbb{F})\boldsymbol{n}]_\tau = \boldsymbol{h} & \textit{on } \Gamma. \end{cases} \tag{0.15}$$

Idea of the proof :

Since Ω is not axisymmetric, we have the uniform bound of $(\boldsymbol{u}_\alpha, \pi_\alpha)$ in $\mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ with respect to α for all $p \in (1, \infty)$. Then there exists $(\boldsymbol{u}_0, \pi_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ such that as $\alpha \to 0$ in $L^{t(p)}(\Gamma)$,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_0, \pi_0) \quad \text{weakly in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega).$$

It can be easily shown that $(\boldsymbol{u}_0, \pi_0)$ is a solution of the Stokes problem (0.15) with Navier boundary condition, corresponding to $\alpha = 0$.

Indeed, passing limit in the variational formulation satisfied by $(\boldsymbol{u}_\alpha, \pi_\alpha)$

$$2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma$$

and as $\alpha \boldsymbol{u}_{\alpha\tau}$ is bounded and goes to $\boldsymbol{0}$ weakly, we get the weak formulation of (0.15).

Now taking difference between the original system (0.14) and the limiting system (0.15), we get,

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) + \nabla(\pi_\alpha - \pi_0) = \mathbf{0}, \ \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0) = 0 & \text{in } \Omega, \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_0) \cdot \boldsymbol{n} = 0, \ 2\left[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)\boldsymbol{n}\right]_\tau + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_0)_\tau = -\alpha \boldsymbol{u}_{0\tau} & \text{on } \Gamma. \end{cases}$$

Once again using the usual L^p-estimate for the above system and also using Hölder inequality and trace theorem, we obtain

$$\begin{aligned} \|\boldsymbol{u}_\alpha - \boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi_\alpha - \pi_0\|_{L^p(\Omega)} &\leq C_p(\Omega) \ \|\alpha \boldsymbol{u}_{0\tau}\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)} \\ &\leq C_p(\Omega) \ \|\alpha\|_{L^{t(p)}(\Gamma)} \|\boldsymbol{u}_0\|_{\mathbf{W}^{1,p}(\Omega)} \end{aligned}$$

which shows that $\boldsymbol{u}_\alpha - \boldsymbol{u}_0$ and $\pi_\alpha - \pi_0$ both tend to zero in the same rate as α.

Theorem (α tends to ∞)

Let $p \in (1, \infty)$ and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the solution of (S) where $\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $\boldsymbol{h} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and α a constant.

i) *Then as $\alpha \to \infty$, we have the convergence,*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ is the unique solution of the Stokes problem with Dirichlet boundary condition,

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\infty = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

ii) *Moreover, for $p \geq 2$, we obtain the strong convergence*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega).$$

Idea of the proof :

The proof is very much similar to the previous theorem. Since $\alpha \to \infty$, we can consider $\alpha \geq 1$ and then we have the bound of $(\boldsymbol{u}_\alpha, \pi_\alpha)$ in $\mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ with respect to α, hence there exists $(\boldsymbol{u}_\infty, \pi_\infty) \in \mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$ such that

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \rightharpoonup (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{weakly in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega).$$

The first key point is to write the system in the following way

$$\begin{cases} -\Delta \boldsymbol{u}_\alpha + \nabla \pi_\alpha = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\alpha = 0 & \text{in } \Omega \\ \boldsymbol{u}_\alpha = \frac{1}{\alpha}\left(\boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\alpha)\boldsymbol{n}]_\tau\right) & \text{on } \Gamma \end{cases}$$

where passing limit as $\alpha \to \infty$, we get that $(\boldsymbol{u}_\infty, \pi_\infty)$ is indeed a solution of the Dirichlet problem.

Now for the strong convergence, taking difference between the two systems, we get

$$\begin{cases} -\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) + \nabla(\pi_\alpha - \pi_\infty) = \mathbf{0}, \ \operatorname{div}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) = 0 & \text{in } \Omega \\ (\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \cdot \boldsymbol{n} = 0, \ 2\,[\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\boldsymbol{n}]_\tau + \alpha(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_\tau = \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_\tau & \text{on } \Gamma \end{cases}$$

and thus the relation

$$2\int_\Omega |\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)|^2 + \alpha \int_\Gamma |(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)_\tau|^2$$
$$= \langle \boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_\tau, \boldsymbol{u}_\alpha - \boldsymbol{u}_\infty \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)}$$

which shows the strong convergence of $\boldsymbol{u}_\alpha$ to $\boldsymbol{u}_\infty$ in $\boldsymbol{H}^1(\Omega)$.
And the strong convergence for the pressure term follows from the following estimate

$$\|\pi_\alpha - \pi_\infty\|_{L^2} \leq \|\nabla(\pi_\alpha - \pi_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)} \leq C\|\Delta(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

For $p > 2$, as $\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \to 0$ in $\mathbf{L}^2(\Omega)$, $\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty) \to 0$ for almost every $\mathbf{x} \in \Omega$ up to a subsequence. And using the uniform estimate of $\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty$, we can write

$$\|\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_p(\Omega)\|\boldsymbol{h} - 2[(\mathbb{D}\boldsymbol{u}_\infty)\boldsymbol{n}]_\tau\|_{\mathbf{W}^{-\frac{1}{p},p}(\Gamma)}$$

which shows that $\|\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\mathbb{L}^p(\Omega)} \leq C$. Therefore, dominated convergence theorem yields $\|\mathbb{D}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\infty)\|_{\mathbb{L}^p(\Omega)} \to 0$. This completes the proof.

Theorem (α less regular)

Let $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\Gamma)$ and $\alpha \in L^{\frac{4}{3}}(\Gamma)$. Then the Stokes problem (S) has a solution $(\boldsymbol{u}, \pi)$ in $\boldsymbol{H}^1(\Omega) \times L^2_0(\Omega)$.

The above result can be proved using the density of $\mathcal{D}(\Gamma)$ in $L^{\frac{4}{3}}(\Gamma)$ and from the good estimates (independent of α) in $\boldsymbol{H}^1(\Omega)$.

VI. Non-linear problem

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \left[(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}\right]_\tau + \alpha\, \boldsymbol{u}_\tau = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \qquad \text{(NS)}$$

Let $1 < p < \infty$ and

$$\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega), \quad \mathbb{F} \in \mathbb{L}^p(\Omega), \quad \boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$$

with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on Γ and $\alpha \in L^{t(p)}(\Gamma)$. Then $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfies (NS) in the sense of distribution is equivalent to:
$\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ such that for all $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}_{\sigma,\tau}(\Omega)$,

$$2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) + \int_\Gamma \alpha \boldsymbol{u}_\tau \cdot \boldsymbol{\varphi}_\tau = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega \mathbb{F} : \nabla \boldsymbol{\varphi} + \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_\Gamma$$

where $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \boldsymbol{w}$.

The operator b satisfies the usual properties:

Lemma

The trilinear form b is defined and continuous on $\boldsymbol{H}^1_{\sigma,\tau}(\Omega) \times \boldsymbol{H}^1_{\sigma,\tau}(\Omega) \times \boldsymbol{H}^1_{\sigma,\tau}(\Omega)$. *Also, we have*

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = \mathbf{0} \tag{0.16}$$

and

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1_{\sigma,\tau}(\Omega).$$

Also note that if Ω *is axisymmetric, we have*

$$b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\beta}) = 0 \quad \text{and} \quad b(\boldsymbol{\beta}, \boldsymbol{\beta}, \boldsymbol{u}) = 0.$$

Now we give the existence and estimate of generalized solution of the Navier-Stokes problem (NS).

Theorem

Let $p > \frac{3}{2}$ *and* $\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega)$, $\boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma)$, $\alpha \in L^{t(p)}(\Gamma)$. *Then the problem* (NS) *has a solution* $(\boldsymbol{u}, \pi)$ *belonging to* $\mathbf{W}^{1,p}(\Omega) \times L^p_0(\Omega)$.

Estimate.

In the Hilbert case, we have the following estimates as in the linear problem:

i) if Ω is not axisymmetric, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right)$$

ii) if Ω is axisymmetric and

- $\alpha \geq \alpha_* > 0$ on $\Gamma_0 \subseteq \Gamma$, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \frac{C(\Omega)}{\min\{2, \alpha_*\}}\left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right)$$

- $\boldsymbol{f}, \mathbb{F}, \boldsymbol{h}$ satisfy the condition $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\beta} - \int_\Omega \mathbb{F} : \nabla\boldsymbol{\beta} + \langle \boldsymbol{h}, \boldsymbol{\beta}\rangle_\Gamma = 0$, then the solution $\boldsymbol{u}$ satisfies $\int_\Gamma \alpha\boldsymbol{u}\cdot\boldsymbol{\beta} = 0$ and

$$\|\mathbb{D}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \int_\Gamma \alpha|\boldsymbol{u}_\tau|^2 \leq C(\Omega)\left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right)^2.$$

In particular, if α is a constant, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\Omega)\left(\|\boldsymbol{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right).$$

Idea of the proof.

- $p = 2$. The existence of solution can be shown using standard arguments applying Galerkin method and Brower's fixed point theorem.
- $p > 2$. Existence of solution follows from bootstrap argument using Hilbert case, exploiting the regularity of the non-linear term.
- $p \in (\frac{3}{2}, 2)$. Here we use the same argument as done by D. Serre for the Dirichlet problem.
- And the estimates follows from the weak formulation exactly by the same argument as in the linear problem due to the properties of the trilinear form b.

Finally we give some results of the limiting problems corresponding to the non-linear system.

Theorem (α tends to 0)

Let $p \geq 2$, Ω be not axisymmetric and $(\boldsymbol{u}_\alpha, \pi_\alpha)$ be the solution of (NS) *where*

$$\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \mathbf{W}^{-\frac{1}{p},p}(\Gamma) \text{ and } \alpha \in L^{t(p)}(\Gamma).$$

Then we have the convergence,

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_0, \pi_0) \quad in \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega) \quad as \quad \alpha \to 0 \quad in \quad L^{t(p)}(\Gamma)$$

where $(\boldsymbol{u}_0, \pi_0)$ is a solution of the following Navier-Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \boldsymbol{u} = 0 & in \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [(2\mathbb{D}\boldsymbol{u} + \mathbb{F})\boldsymbol{n}]_\tau = \boldsymbol{h} & on \quad \Gamma. \end{cases}$$

Theorem (α tends to ∞)

Let $p \geq 2$ *and* $(\boldsymbol{u}_\alpha, \pi_\alpha)$ *be the solution of* (NS) *where*

$$\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega), \mathbb{F} \in \mathbb{L}^p(\Omega), \boldsymbol{h} \in \boldsymbol{H}^{-1/2}(\Gamma) \text{ and } \alpha \text{ is a constant .}$$

Then as $\alpha \to \infty$*, we have the convergence,*

$$(\boldsymbol{u}_\alpha, \pi_\alpha) \to (\boldsymbol{u}_\infty, \pi_\infty) \quad \text{in} \quad \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$$

where $(\boldsymbol{u}_\infty, \pi_\infty)$ *is a solution of the Navier-Stokes problem with Dirichlet boundary condition,*

$$\begin{cases} -\Delta \boldsymbol{u}_\infty + \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty + \nabla \pi_\infty = \boldsymbol{f} + \operatorname{div} \mathbb{F} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega \\ \boldsymbol{u}_\infty = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Thank You