Control of Some Infinite Networks

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The setting

We consider the following problem :

- the wave equation : $(\partial_t^2 - \partial_x^2)u = 0, \quad u = u(x,t).$
- the domain : $t \geq 0$, $x \in \mathcal{N}$ where $\mathcal{N}$ is a **infinite** network (unbounded).
- boundary conditions : Dirichlet.
- vertex conditions : continuity & damping (or Kirchhoff).
- initial conditions : $u(x,0) = u^0(x), \partial_t u(x,0) = u^1(x).$

The questions

The goal :

What about the energy (local or global) of the solution?
Can a finite-time Stabilization occur?

- energy decay when $t \to +\infty$?
- if any, is it exponential: $e^{-\gamma t}, \gamma > 0$?
- what is the decay rate $\gamma =$?
- is it optimal?
- what is the limit energy of the system?

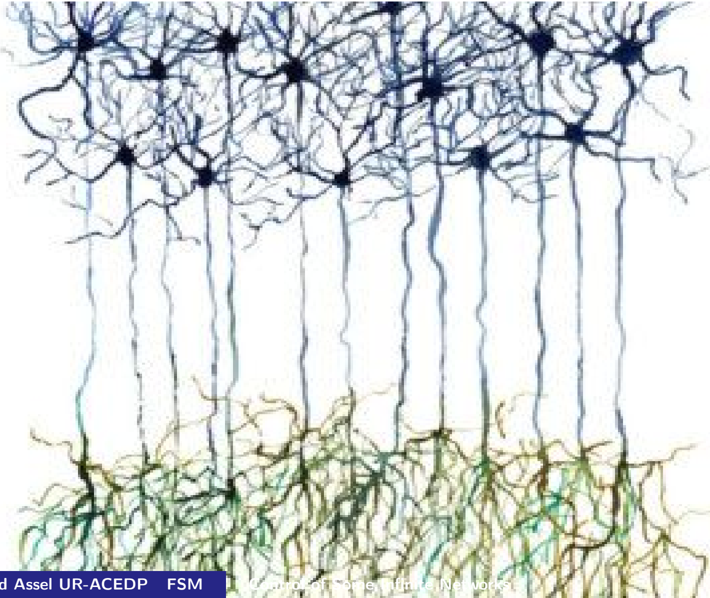
The model

The model : a network with one infinite edge:

$$\mathcal{N} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{U}$$

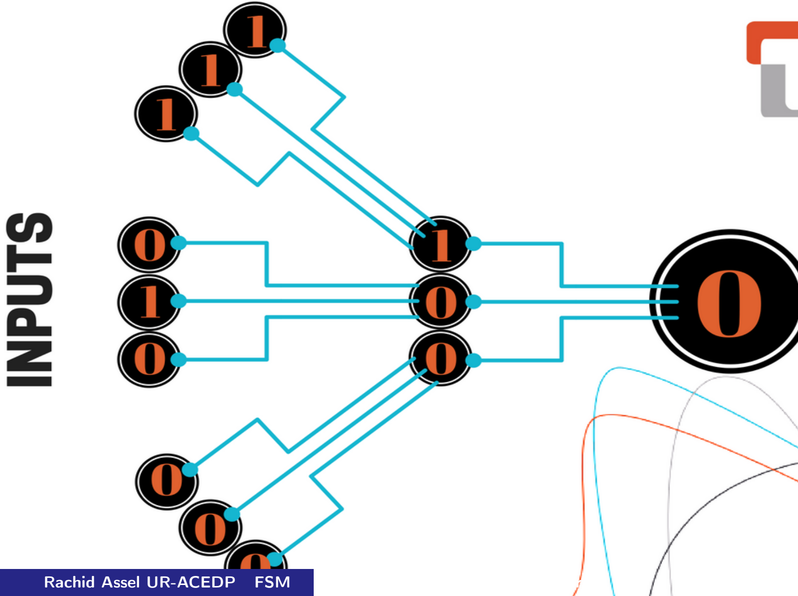
- $\mathcal{V} = \{0\}$: a single vertex identified to 0,
- $\mathcal{E} = \{e_j\}_{j=1,\dots,N \geq 1}$: finite edges $e_1, \dots, e_N$, identified to the intervals $[0, \ell_j], \ell_j > 0, j = 1, \dots, N$
- $\mathcal{U} = \{e_0\}$: a single infinite edge identified to $[0, +\infty[$.

Neural network

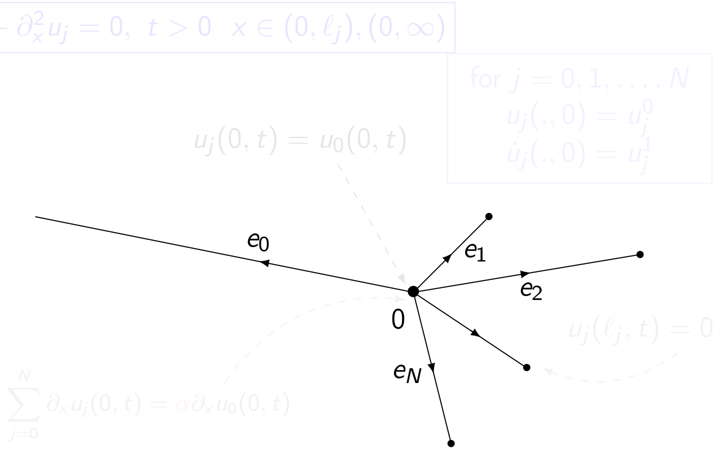


Neural network

Artificial Neural network



Schematic description of the situation



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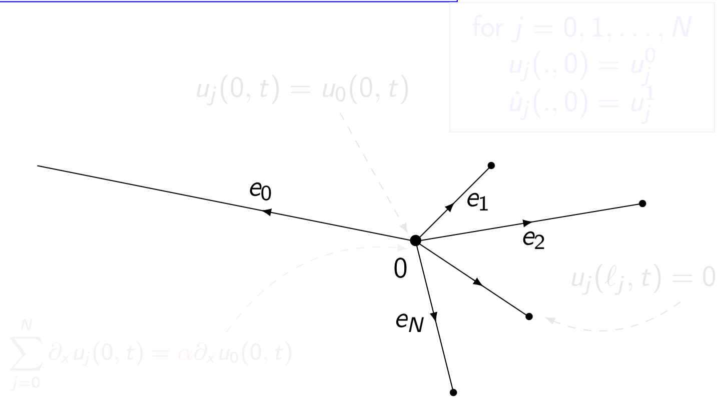
$$\partial_t^2 u_j - \partial_x^2 u_j = 0, \;\; t > 0 \;\; x \in (0, \ell_j), (0, \infty)$$



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The setting

Case without damping: $\alpha = 0$

Case with damping: $\alpha > 0$

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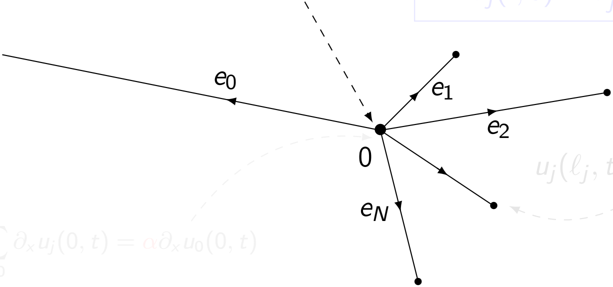
for $j = 0, 1, \dots, N$
$u_j(., 0) = u_j^0$
$\dot{u}_j(., 0) = u_j^1$

$$u_j(0, t) = u_0(0, t)$$



$$u_j(\ell_j, t) = 0$$

$$\sum_{j=0}^{N} \partial_x u_j(0, t) = \alpha \partial_x u_0(0, t)$$

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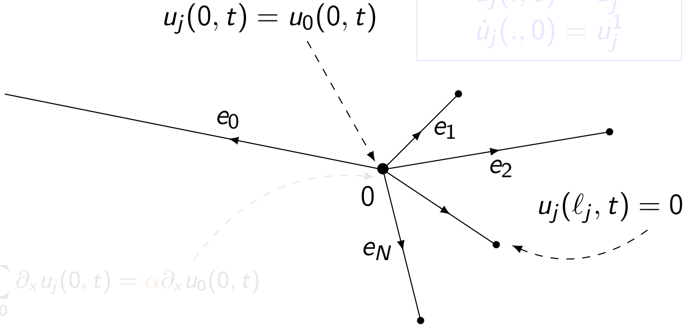

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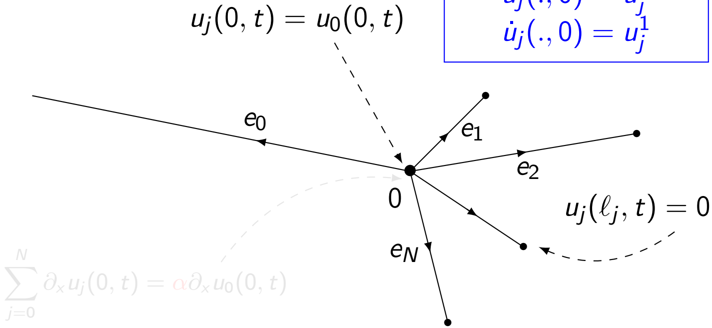

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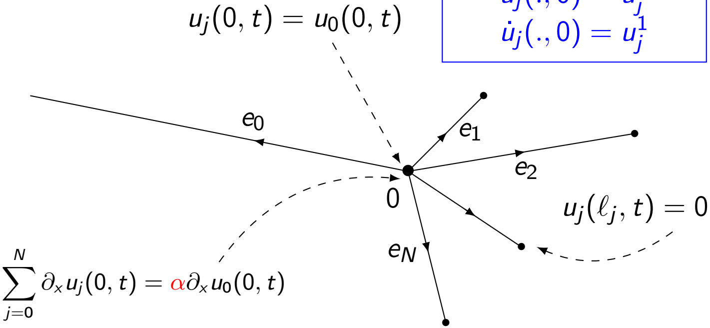

The equation

We want to study the behaviour, for large t, of the energy of the solution $u = (u_0, u_1, \dots, u_N)$ to the system

$$\begin{cases} \partial_t^2 u_j(x,t) - \partial_x^2 u_j(x,t) = 0, \ x \in (0,\ell_j), \ t > 0 \ \text{ et } j \in \{1,\dots,N \\ \partial_t^2 u_0(x,t) - \partial_x^2 u_0(x,t) = 0, \ x \in]0,+\infty[, \ t > 0, \\ u_j(0,t) = u_0(0,t) \ \text{ et } \ u_j(\ell_j,t) = 0, \ t \geq 0, \ \ j \in \{1,\dots,N\} \\ \displaystyle\sum_{j=0}^{N} \frac{\partial u_j}{\partial x}(0,t) = \alpha \frac{\partial u_0}{\partial x}(0,t) = 0, \ t \geq 0 \quad \text{(DC,KC)} \\ u_j(.,0) = u_j^0, \ \partial_t u_j(.,0) = u_j^1 \text{ on } (0,\ell_j) \text{ for } j \in \{1,\dots,N\} \\ u_0(.,0) = u_0^0, \ \partial_t u_0(.,0) = u_0^1 \text{ on } (0,+\infty) \end{cases} \tag{1.1}$$

α is a real constant : it is a damping factor if it is positive.

Wellposedness

Consider the functional spaces

$$V = \left\{ \Phi \in \dot{H}^1(0,+\infty) \times \prod_{j=1}^{N} H^1(0,l_j) \ ; \ \varphi_j(0) = \varphi_0(0) \text{ et } \varphi_j(l_j) = 0, 1 \leq j \leq N \right.$$

where $\Phi = (\varphi_0, \varphi_1, \ldots, \varphi_N)$ and

$$\dot{H}^1(0,+\infty) = \overline{C_0^\infty\left([0,+\infty[\right)}, \text{ for the norm} \|\varphi\|_{\dot{H}^1} = \|\varphi'\|_{L^2(0,+\infty)}.$$

V is a Hilbert space when equipped with the inner product :

$$\langle \Phi | \Psi \rangle = \sum_{j=1}^{N} \langle \varphi_j | \psi_j \rangle_{L^2(0,l_j)} + \langle \varphi_0 | \psi_0 \rangle_{L^2(0,+\infty)}$$

Wellposedness

$$X = L^2(\mathcal{G}) = \left(L^2(0,+\infty) \times \prod_{j=1}^{N} L^2(0,l_j) \right)$$

If we let $A = \begin{pmatrix} 0 & I \\ \Delta_{\mathcal{N}} & 0 \end{pmatrix}$, with $\Delta_{\mathcal{N}}$ is the Laplacien on $\mathcal{N}$ with Dirichlet boundary conditions, continuity et damping in the vertex, the system (1.1) writes as Cauchy problem

$$\begin{cases} \dot{\Psi}(t) = A\Psi(t) \\ \Psi(0) = (f,g) \end{cases} \tag{1.2}$$

where $\Psi = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$ and $(f,g) \in V \times X$ are given in terms of the initial conditions (u^0, u^1).

Wellposedness

Stone's Theorem $\Rightarrow$ the operator A generates a C_0-semigroup .

$$\Downarrow$$

By the semi-group theory $\Rightarrow$ the problem is well-posed and the system has a unique solution u such that

$$(u, \partial_t u) \in \mathcal{C}^0\left((0, +\infty), V\right) \cap \mathcal{C}^1\left((0, +\infty), X\right).$$

The energy

Let $R \in]0, +\infty]$. If u is the solution of (1.1) then its energy is given by :

$$E_R(u)(t) = \frac{1}{2}\sum_{j=1}^{N} \|\partial_t u_j\|^2_{L^2(0,l_j)} + \|\partial_x u_j\|^2_{L^2(0,l_j)} + \frac{1}{2}\left(\|\partial_t u_0\|^2_{L^2(0,R)} + \|\partial_x u_0\|^2_{L^2(0,R)}\right)$$

When $R = +\infty$, $E_\infty(u)$ is the global energy of u.
When $R < +\infty$, $E_R(u)$ is the local energy u.
We ask the question on the behaviour of $E_R(u)(t)$ as t becomes large.

$$\forall t > 0, \quad E_\infty(u)(t) - E_\infty(u)(0) = -\alpha \int_0^t |\partial_s u_0(0,s)|^2 ds. \qquad (1.3)$$

Case $\alpha = 0$ et $\ell_j = \ell > 0, \forall j = 1, \dots, N$

In this case, the global energy is conserved. So we focus on the local energy

Theorem (AJKh, JDE 2017)

Assume that $\alpha = 0$ *et* $\ell_j = \ell > 0, \forall j = 1, \dots, N$ *For* $2\ell < R < +\infty$, *we have*

$$|E_R(u) - E^\infty| \leq Ce^{-\gamma t}, \quad \text{for large } t$$

with $\gamma = \dfrac{1}{\ell}\mathrm{Arcoth(N)}$ *and* E^∞ *is a limit energy explicitly given in terms of the initial data.*

Case $\alpha = 0$ et $\frac{\ell_j}{\ell_k} \notin \mathbb{Q}, \forall j, k = 1, \ldots, N$

Assume that $\alpha = 0$ and $\frac{\ell_j}{\ell_k} \notin \mathbb{Q}, \forall j, k = 1, \ldots, N$.

Theorem (AKhR)

There exists a sequence $(\lambda_n)_n$ of eigenvalues of A such that $\Im(\lambda_n) \to \infty$ and $\Re(\lambda_n) \to 0$.

$$\Downarrow$$

There is no exponential stabilization.

Outline of the proof : case $N = 2$.

- we start by the rational case $\frac{\ell_2}{\ell_1} = \frac{p}{q}, p \wedge q = 1$.
  λ is an eigenvalue of A iff $Z = e^{\frac{2\pi\ell}{q}\lambda}$ is a root of
  $$Q(Z) = 3Z^{p+q} - Z^p - Z^q - 1.$$
  The real root $Z = 1$ generates a sequence of sequence of eigenvalues $\lambda_k = i\frac{k\pi q}{l}, k \in \mathbb{Z}^*$.
- for the irrational case : we let $d = \frac{\ell_2}{\ell} \notin \mathbb{Q}, (\ell = \ell_1)$. We can approximate d by a rational sequence with a prescribed precision: i.e. there exists a sequence $(p_n, q_n) \in \mathbb{N}^* \times \mathbb{N}^*, \quad p_n \wedge q_n = 1$ such that
  $$\left| d - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}, \forall n \in \mathbb{N}.$$

Let $\mu = 2\pi\lambda\ell$. The equation for μ is

$$3e^{(1+d)\mu} - e^{d\mu} - e^{\mu} - 1 = 0.$$

Writing : $d = [d] + \dfrac{p_n}{q_n} + \dfrac{r_n}{q_n^2}$ with $|r_n| \leq 1, \forall n \in \mathbb{N}$ and the equation for μ in the form $f(\mu) + g(\mu) = 0$ with

$$f(\mu) = 3e^{\left(1+[d]+\frac{p_n}{q_n}\right)\mu} - e^{\left([d]+\frac{p_n}{q_n}\right)\mu} - e^{\mu} - 1$$

and

$$g(\mu) = \left(e^{\frac{r_n}{q_n^2}\mu} - 1\right)\left(3e^{\left(1+[d]+\frac{p_n}{q_n}\right)\mu} - e^{\left([d]+\frac{p_n}{q_n}\right)\mu}\right).$$

The function f corresponds to the rational case :

$$f(\mu_{n,k}) := f(2k\pi q_n) = 0, k \in \mathbb{N}.$$

Using the Rouché theorem, we prove the existence of zeroes of $f + g$ in small neighbourhoods of $\mu_{n,k}$.

Case with damping: $\alpha > 0$

We assume that $\alpha > 0$.
The global energy is decreasing.
The spectrum : $\sigma(A) = \sigma_{ac}(A) \cup \sigma_p(A) =]-\infty, 0] \cup \sigma_p(A)$.

$$\lambda \in \sigma_p(A) \quad \text{iff} \quad \mathrm{e}^{2\lambda\ell} = \frac{\alpha - (\mathrm{N}+1)}{\alpha + (\mathrm{N}-1)}.$$

There are cases: discuss the sign of $\alpha - (N+1)$.

Theorem (AGh, EECT 2018)

We assume that $\ell_j = \ell > 0, \forall j = 1, \dots, N$ *and we let* $r = \frac{1-\alpha}{N}$.

1. $\forall \alpha > 0,\ \lambda_n := \frac{in\pi}{\ell}, n \in \mathbb{Z}^*$ *is a simple eigenvalue of A.*
2. *For* $\alpha = N + 1$, $\quad \sigma_p(A) = \{\lambda_n, n \in \mathbb{Z}^*\}$
3. *For* $\alpha > N + 1$,
$$\sigma_p(A) = \{\lambda_n, n \in \mathbb{Z}^*\} \cup \left\{ \frac{1}{2\ell} \ln\left(\frac{r+1}{r-1}\right) + \frac{in\pi}{\ell}, n \in \mathbb{Z} \right\}.$$
4. *For* $\alpha < N + 1$,
$$\sigma_p(A) = \{\lambda_n, n \in \mathbb{Z}^*\} \cup \left\{ \frac{1}{2\ell} \ln\left(\frac{1+r}{1-r}\right) + \frac{i(2n+1)\pi}{\ell}, n \in \mathbb{Z} \right\}$$

Theorem (AGh, EECT 2018)

For $\alpha \neq N+1$, the system is exponentially stable. The exponential decay rate is :

1. $\gamma^+ = -\frac{1}{\ell}\mathrm{Arcoth(r)}$ *when* $\alpha > N+1$.
2. $\gamma^- = -\frac{1}{\ell}\mathrm{Argcoth(r)}$ *when* $\alpha < N+1$.

Outline of the proof : a resolvent estimate and contour deformation technique in order to use the residue theorem. Then, modulo some projections on the spaces spanned by the eigenvectors, the semi-group is exponentially decaying.

Case $\alpha = N + 1$: Finite time stabilization

Theorem

We assume that $\alpha = N + 1$. Then , pour $t \geq 2\ell$, the energy is constant : $E(u)(t) = E(u)(2\ell)$. Moreover, for initial data chosen in $(I - \Pi_{pp})X$ where $\Pi_{pp}$is the projection on the subspace spanned by the the eigenvectors of A the energy of the solution is 0 for $t \geq 2\ell$.

Outline of the proof :

- we compute explicitly the resolvent $R(\lambda), \lambda \in \rho(A)$,
- we prove that $R(\lambda)$ is of 2ℓ-exponential type : $\|R(\lambda)\| \leq Ce^{2\ell|\lambda|}, \forall \lambda \in \mathbb{C}$ on the subspace $(I - \Pi_{pp})X$.
- Using a theorem by Xu, the semi-group is super-stable : $U(t) = 0$, pour $t \geq 2\ell$ on $(I - \Pi_{pp})X$.

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Thanks ☺