Homogenization of an elastic medium having three phases

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Introduction

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Introduction

- The elastic behavior of fibrous composites.
- The existence of an intermediate layer between the matrix and the fiber. These layers are called interphases between the fibers and matrix.
- These interphases are formed due to, for example, chemical reaction between the matrix and fiber materials or the use of protective coatings on the fiber during manufacturing.
- The influence of fiber coatings in the overall behavior of fibrous composites.

Introduction

- The interphase can significantly affect the overall mechanical properties of the composites:

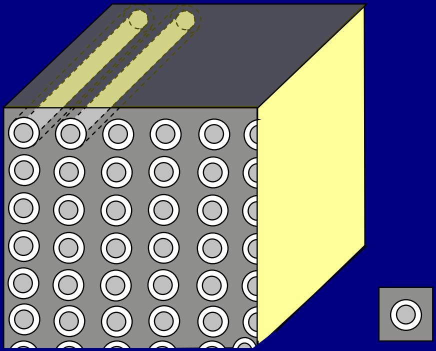
Introduction

- if the interphase material is not stiff enough, the resulting overall material properties are significantly affected especially across the transverse directions (see experimental results in Kari et al. [2006] and Mikata [1985])

Introduction

- *In Hashin [2002]*:
  a) for very compliant interphase there are significant displacement jumps and negligible traction jumps,
  b) for very stiff interphase the situation is reversed,
  c) for the in between case all jumps may be neglected.

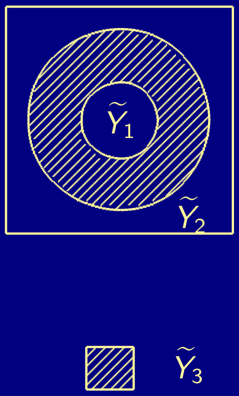
The geometrical model



The geometrical model

- Let $Y =]0,1[^3$ the periodicity cell, we shall assume $Y = \left(\widetilde{Y}_1 \cup \widetilde{Y}_{13} \cup \widetilde{Y}_2 \cup \widetilde{Y}_{23} \cup \widetilde{Y}_3\right) \times]0,1[$ where $\widetilde{Y}_1 \times]0,1[$ is the fiber, $\widetilde{Y}_2 \times]0,1[$ is the coating, $\widetilde{Y}_3 \times]0,1[$ is the matrix and $\widetilde{Y}_{\alpha 3},\ \alpha = 1,2$ are the interfaces between $\widetilde{Y}_\alpha$ and $\widetilde{Y}_3$ (see fig.).

The geometrical model



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  - $\Omega := \widetilde{\Omega} \times \mathrm{I}$, $\mathrm{I} :=]0, 1[$,
  - for $i = 1, 2, 3$, $\widetilde{\Omega}_i^\varepsilon := \{\widetilde{x} \in \widetilde{\Omega};\ \chi_i(\frac{\widetilde{x}}{\varepsilon}) = 1\}$, where χ_i is the characteristic function of $\widetilde{Y}_i$ and $\Omega_i^\varepsilon := \widetilde{\Omega}_i^\varepsilon \times \mathrm{I}$.

Some preliminaries

Given a tensor $A = (a_{ijkl})_{i,j,k,l=1,2,3}$, we shall introduce a planar representation of A:

$$A = (A_{jl})_{j,l=1,2,3} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where, for fixed j and l, $A_{jl} = (a_{ijkl})_{i,k=1,2,3}$.

Some preliminaries

We will further arrange our planar representations on a block form as follows :

$$A = \left(\begin{array}{c|c} \widetilde{A} & C \\ \hline L & A_{33} \end{array} \right)$$

where the blocks are given by

$$\widetilde{A} = (a_{i\beta k\delta}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}, \quad C = (a_{i\beta k3}) = \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix},$$

$$L = (a_{i3k\delta}) = \begin{pmatrix} A_{31} & A_{32} \end{pmatrix}.$$

Some preliminaries

The multiplication of tensors $A = (a_{ijkl})$ and $A' = (a'_{ijkl})$ becomes an ordinary matrix multiplication, namely

$$AA' = \left(\begin{array}{c|c} \widetilde{A} & C \\ \hline L & A_{33} \end{array}\right) \left(\begin{array}{c|c} \widetilde{A}' & C' \\ \hline L' & A'_{33} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \widetilde{A}\,\widetilde{A}' + CL' & \widetilde{A}C' + CA'_{33} \\ \hline L\widetilde{A}' + A_{33}L' & LC' + A_{33}A'_{33} \end{array}\right).$$

Some preliminaries

Given a matrix $\xi = (\xi_{ij})_{i,j=1,2,3}$, we shall introduce a linear representation of ξ by considering it as a column vector in j. Writing it on a block form:

$$\xi = \begin{pmatrix} \widetilde{\xi} \\ \xi_3 \end{pmatrix}$$

with $\widetilde{\xi} = (\xi_{i\beta})\,,\ \xi_3 = (\xi_{i3})\,.$

Some preliminaries

If $A = (a_{ijkl})$ is a $3 \times 3 \times 3 \times 3$ tensor and $\xi = (\xi_{ij})$ a 3×3 matrix, the product $A\xi$ can be done in the block representation as :

$$A\xi = \left(\begin{array}{c|c} \widetilde{A} & C \\ \hline L & A_{33} \end{array} \right) \left(\begin{array}{c} \widetilde{\xi} \\ \xi_3 \end{array} \right) = \left(\begin{array}{c} \widetilde{A}\widetilde{\xi} + C\xi_3 \\ L\widetilde{\xi} + A_{33}\xi_3 \end{array} \right).$$

Some preliminaries

Scalar product between matrices considered as vectors is denoted by $[.,.]$:

$$[\xi,\eta] := \xi_{ij}\eta_{ij} = \xi_{i\beta}\eta_{i\beta} + \xi_{i3}\eta_{i3} = [\widetilde{\xi},\widetilde{\eta}] + \xi_3\eta_3.$$

However, the ordinary scalar product in $\mathbb{R}^3$ is denoted "*dot*". In both cases the corresponding norms are denoted $|.|$.

Some preliminaries

Given any $\phi \in (\mathcal{D}'(\Omega))^3$, we denote $\mathrm{e}(\phi) := \mathrm{sym}\nabla\phi = \frac{1}{2}\left(\nabla\phi + \nabla\phi^T\right)$, i.e. the symmetric tensor of $(\mathcal{D}'(\Omega))^9_s$, the components of which are given by :

Some preliminaries

$$e_{ij}(\phi) = \frac{1}{2}\left(\partial_i\phi_j + \partial_j\phi_i\right), i, j = 1, 2, 3.$$

We shall also use the following abbreviated notation

$$e(\phi) = \left(\begin{array}{c|c} e_{\alpha\beta}(\phi) & e_{\alpha 3}(\phi) \\ \hline e_{\alpha 3}(\phi) & e_{33}(\phi) \end{array}\right), \ \alpha, \beta = 1, 2.$$

The physical setting

For $m = 1, 2, 3$, let $\mathbb{A}^m = \left(a^m_{jikl}\right)$ be the stiffness tensor of the mth component. They satisfy the usual symmetry and coerciveness assumptions :

- $a^m_{ijkl} = a^m_{jikl} = a^m_{ijlk} = a^m_{klij}$
- $[\mathbb{A}^m \xi, \xi] = a^m_{ijkl}\xi_{ij}\xi_{kl} \geq C_0 \left|\xi\right|^2, C_0 > 0,\ \xi \in \mathbb{R}^9_s.$

Furthermore, $\mathbb{A}^1$ has the following block form :

$$\mathbb{A}^1 = \left(\begin{array}{c|c} \widetilde{A}^1 & C^1 \\ \hline L^1 & A^1_{33} \end{array}\right) = \left(\begin{array}{c|c} a^1_{i\beta k\delta} & a^1_{i\beta k3} \\ \hline a^1_{i3k\delta} & a^1_{i3k3} \end{array}\right).$$

The physical setting

Let

$$\mathbb{A}^{\varepsilon}(x) = \chi_1^{\varepsilon}(x)\mathbb{A}^{1,\varepsilon} + \chi_2^{\varepsilon}(x)\mathbb{A}^{2} + \varepsilon^{2}\chi_3^{\varepsilon}(x)\mathbb{A}^{3}$$

where

$$\mathbb{A}^{1,\varepsilon} = \left(\begin{array}{c|c} \varepsilon^{2}\widetilde{A}^{1} & \varepsilon C^{1} \\ \hline \varepsilon L^{1} & A_{33}^{1} \end{array}\right)$$

and

$$\chi_m^{\varepsilon}(x) = \chi_m(\frac{x}{\varepsilon}), \;\; x \in \Omega, \;\; m = 1, 2, 3.$$

Remark

The term $\varepsilon^2 \widetilde{A}^1$ should be interpreted as a possible rotation of the "principal axes" of the elasticity tensor in the transverse plane, and the terms εL^1 and εC^1 correspond to a possible slightly deviation of the main axes of anisotropy from the x_3-direction. Moreover, for any $\xi = (\xi_{ij}) \in \mathbb{R}^9_s$ we deduce

$$\left[\mathbb{A}^{1,\varepsilon}\xi, \xi\right] = \widetilde{A^1}\left(\varepsilon\widetilde{\xi}\right).\left(\varepsilon\widetilde{\xi}\right) + 2L^1\left(\varepsilon\widetilde{\xi}\right).\xi_3 + A^1_{33}\xi_3.\xi_3.$$

Remark

By virtue of

$$A^1_{33}\eta.\eta = a^1_{i3k3}\eta_i\eta_k = a^1_{ijkl}\eta_i\delta_{j3}\eta_k\delta_{l3} \geq C_0\,|\eta|^2 \text{ for } \eta \in \mathbb{R}^3,$$

where δ_{mp} is the Kronecker symbol, the tensor $\mathbb{A}^{1,\varepsilon}$ satisfies the following "relaxed" anisotropic ellipticity condition :

$$\mathbb{A}^{1,\varepsilon}_{ijkl}\xi_{ij}\xi_{kl} \geq C_0\left[\varepsilon^2\left|\widetilde{\xi}\right|^2 + |\xi_3|^2\right], \text{ for any } \xi = (\xi_{ij}) \in \mathbb{R}^9_s.$$

Mathematical problem

In the presence of a given body force $F^\varepsilon \in L^2(\Omega; \mathbb{R}^3)$, the displacement field u^ε of the composite, which occupies Ω, at equilibrium, is then the solution of the following problem

Mathematical problem

$$(\mathcal{P}_\varepsilon)\begin{cases} -\mathrm{div}_x\left(\mathbb{A}^\varepsilon(x)\mathrm{e}(u^\varepsilon)\right) = F^\varepsilon(x) \quad \text{in } \Omega, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega \cap \{-\frac{1}{2} < x_3 < \frac{1}{2}\} =: \Gamma_L, \\ \mathbb{A}^\varepsilon(x)\mathrm{e}(u^\varepsilon).n = 0 \quad \text{on } \partial\Omega \setminus \Gamma_L, \end{cases}$$

where n denotes the outer normal to Ω, the subscript L stands for lateral boundary.

Variational formulation of $(\mathcal{P}_\varepsilon)$

For every $\varepsilon > 0$, this problem admits a unique weak solution u^ε belonging to the space

$$H^1_L(\Omega;\mathbb{R}^3) := \left\{u \in H^1(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \Gamma_L\right\}.$$

The function u^ε is then the unique solution of the following system:

Variational formulation of $(\mathcal{P}_\varepsilon)$

$$\begin{cases} u^\varepsilon \in H^1_L(\Omega;\mathbb{R}^3), \\ \int_\Omega [\mathbb{A}^\varepsilon(x)\mathrm{e}(u^\varepsilon),\mathrm{e}(v)]\,dx = \int_\Omega F^\varepsilon v dx, \ \forall v \in H^1_L(\Omega;\mathbb{R}^3) \end{cases}$$

A literature review

Homogenization of problems of this kind, in composite media with fibers, has been considered by Bouchité-Bellieud [2002],Caillerie [1985],Brillard-Jarroudi [2001],Sili [2005] and further references therein. Most of the previous works dealt with the case of the fiber-reinforced composite materials without fiber coatings.

A literature review

Let us emphasize that the combination of the fiber coating softness, the high anisotropy of the fiber is an interesting challenge in the homogenization process. In particular, we will see that the resulting two-scale homogenized systems are "strongly" influenced by this combination : the effective displacement field is obtained by solving a homogenized problem in the domain Ω and an auxiliary problem in the coated fiber $\overline{Y_1 \cup Y_3}$ with a non-standard boundary condition across the transverse directions

A literature review

One of the aims of the present work is to provide "rigorous" models for elastic materials having three phases : matrix, fibers and fiber coatings. In particular, we derive some new models which describe the interaction between the mechanical processes in the fibers and in the fiber coatings and recover a similar effects, as in Hashin, of soft/stiff interphase on the overall behavior of three-phases fiber composites.

A literature review

As far as we know, the closest work, from a mathematical point of view, to ours was done by the author in 2009, where a heat transfer problem was studied in the same geometrical framework. Thus, the present study appears as a natural extension to the linear elasticity system.

A priori estimates

Lemma

There exists $C > 0$, independent of ε, s.t.

$$\|u^\varepsilon\|_\Omega \leq C,$$

$$\left(\varepsilon \, \|\widetilde{\mathrm{e}}(u^\varepsilon)\|_{\Omega_1^\varepsilon}, \|\mathrm{e}_3(u^\varepsilon)\|_{\Omega_1^\varepsilon}\right) \leq C,$$

$$\|\mathrm{e}(u^\varepsilon)\|_{\Omega_2^\varepsilon} \leq C,$$

$$\varepsilon \, \|\mathrm{e}(u^\varepsilon)\|_{\Omega_3^\varepsilon} \leq C.$$

Main ingredient of the proof

It is known (see Boughammoura [2012]) that there exists some constant C, which does not depend on ε, such that, for any function $v \in H^1_L(\Omega;\mathbb{R}^3)$ we have :

$$||v||^2_{\Omega} \leq C\left(||\mathrm{e}\,(v)\,||^2_{\Omega_2^\varepsilon} + \varepsilon^2||\mathrm{e}\,(v)\,||^2_{\Omega_1^\varepsilon \cup \Omega_3^\varepsilon}\right).$$

The two-scale convergence

Definition 1

A function $\phi(x,y) \in L^2(\Omega \times Y, \mathbb{R}^3)$ which satisfies

$$\lim_{\varepsilon \to 0} \int_\Omega \left|\phi(x, \frac{x}{\varepsilon})\right|^2 dx = \int_Q \int_Y |\phi(x,y)|^2 \, dxdy.$$

is called admissible test function.

The two-scale convergence

Definition 2

A sequence u^ε in $L^2(\Omega, \mathbb{R}^3)$ is said to two-scale converge to a function $u^0 \in L^2(\Omega \times Y, \mathbb{R}^3)$, and we denote $u^\varepsilon \xrightarrow{2s} u^0$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(t,x)\phi(x, \frac{x}{\varepsilon})dx = \int_Q \int_Y u^0(x,y)\phi(x,y)dxdy$$

for each admissible test function ϕ.

The two-scale convergence

Proposition

Let $\{u^\varepsilon\}$ be a sequence in $L^2(\Omega;\mathbb{R}^3)$.

1. If $\{u^\varepsilon\}$ is bounded in $H^1(\Omega;\mathbb{R}^3)$, then there exists a unique $u \in H^1(\Omega;\mathbb{R}^3)$ and a unique $u_1 \in L^2(\Omega; H^1(Y;\mathbb{R}^3)/\mathbb{R}^3)$ such that, at least for a subsequence,

$$u^\varepsilon \xrightarrow{2s} u, \;\; \mathrm{e}(u^\varepsilon) \xrightarrow{2s} \mathrm{e}_x(u) + \mathrm{e}_y(u_1).$$

The two-scale convergence

Proposition

Let $\{u^\varepsilon\}$ be a sequence in $L^2\left(\Omega;\mathbb{R}^3\right)$.

2. If $\{u^\varepsilon\}$ and $\{\varepsilon \mathrm{e}(u^\varepsilon)\}$ are bounded in $L^2\left(\Omega;\mathbb{R}^3\right)$ and in $L^2\left(\Omega;\mathbb{R}^9\right)$ respectively, then there exists a unique $u_0 \in L^2\left(\Omega; H^1\left(Y;\mathbb{R}^3\right)/\mathbb{R}^3\right)$ such that, at least for a subsequence,

$$u^\varepsilon \xrightarrow{2s} u_0, \ \varepsilon \mathrm{e}(u^\varepsilon) \xrightarrow{2s} \mathrm{e}_y(u_0).$$

The two-scale convergence

Proposition

There exists

$$u_2 \in \left(H^1_L(\Omega)\right)^3,$$

$$v_1^\alpha \in H^1(\mathrm{I}) \cap L^2\left(\Omega; H^1_\sharp\left(\widetilde{Y}_1\right)/\mathbb{R}^3\right), \ \alpha = 1, 2,$$

$$(v_2, v_3) \in \left(L^2\left(\Omega; H^1_\sharp(Y_2)/\mathbb{R}^3\right)\right)^3 \times \left(L^2\left(\Omega; H^1_\sharp(Y_3)/\mathbb{R}^3\right)\right)^3$$

$$Z \in L^2(\Omega \times Y)^3, \ v_1^3 \in H^1_L(\Omega),$$

The two-scale convergence

Proposition

$$u^\varepsilon(x) \xrightarrow{2s} \chi_1(y)v_1(x,\widetilde{y}) + \chi_2(y)u_2(x) + \chi_3(y)v_3(x,y),$$

The two-scale convergence

Proposition

$$\chi_1^\varepsilon\left(u^\varepsilon, \varepsilon \mathrm{e}_{\alpha\beta}(u^\varepsilon), \mathrm{e}_3(u^\varepsilon)\right) \xrightarrow{2s}$$
$$\chi_1(y)\left(v_1(x,\widetilde{y}), \mathrm{e}^{\widetilde{y}}_{\alpha\beta}(v_1)(x,\widetilde{y}), Z(x,y)\right),$$

The two-scale convergence

Proposition

$$\chi_2^\varepsilon\,(u^\varepsilon, \mathrm{e}(u^\varepsilon)) \xrightarrow{2s} \chi_2(y)\,(u_2(x), (\mathrm{e}^x(u_2)(x) + \mathrm{e}^y(v_2)(x,y))),$$

The two-scale convergence

Proposition

$$\chi_3^\varepsilon\left(u^\varepsilon, \varepsilon \mathrm{e}(u^\varepsilon)\right) \xrightarrow{2s} \chi_3(y)\left(v_3(x,y), \mathrm{e}^y(v_3)(x,y)\right),$$

where

$$\left(v_1^\alpha, v_1^3\right) := \left(v_1^1(x,\widetilde{y}), v_1^2(x,\widetilde{y}), v_1^3(x)\right) =: v_1(x,\widetilde{y}),$$

and $\int_{\mathrm{I}} Z(x,y)dy_3 = \mathrm{e}_3^x(v_1)(x,\widetilde{y})$.

The two-scale convergence

Proposition

Moreover, there exists a unique function $w_3 \in \left(L^2\left(\Omega; H^1_\sharp(Y_3)\right)\right)^3$ such that

$$\begin{cases} v_3(x,y) = u_2(x) + w_3(x,y) \text{ in } Y_3, \\ w_3(x,y) = v_1(x,\widetilde{y}) - u_2(x) \text{ on } Y_{13} := \widetilde{Y}_{13} \times \mathrm{I}, \\ w_3(x,y) = 0 \text{ on } Y_{23} := \widetilde{Y}_{23} \times \mathrm{I}, \end{cases}$$

and u^ε converges weakly in $L^2(\Omega;\mathbb{R}^3)$ to the function

$$U(x) = (1-\theta_1)\, u_2(x) + \int_{\widetilde{Y}_1} v_1(x,\widetilde{y})d\widetilde{y} + \int_{Y_3} w_3(x,y)dy.$$

Homogenization results

First, we introduce for $k, h \in \{1, 2, 3\}$ the vector valued functions w_2^{kh}, unique solutions of the following cellular problems:

$$
\textbf{(cell.2)}^{\textbf{kh}} \begin{cases} \mathrm{d}iv_y \mathbb{A}^2 \mathrm{e}^y(w_2^{kh})(y) = 0 \text{ in } Y_2, \\ \mathbb{A}^2[(\mathrm{e}^y(w_2^{kh}) + \delta^{kh}].n_2(y) = 0 \quad \text{on } Y_{23}, \\ w_2^{kh}(y), \mathbb{A}^2 e^y(w_2^{kh})(y).n_2(y)_{|\partial Y_2 \cap \partial Y} \; Y\text{—periodic}, \end{cases}
$$

where $\delta^{kh} = e_k \otimes e_h$ is the symmetric second-rank tensor with components $\delta_{ij}^{kh} = \delta_{ij}\delta_{kh}$.

Homogenization results

we define the fourth-rank symmetric, positive definite tensors $A_{33}^{1,hom}$ and $A^{2,hom}$ by their entries

$$A_{33}^{1,hom} := \theta_1 A_{33}^1,$$

$$A_{ijkh}^{2,hom} := \int_{Y_2} \mathbb{A}_{ijmp}^2 [\mathrm{e}_{mp}^y (w^{2kh})(y) + \delta_{mp}^{kh}] dy.$$

Homogenization results

Now, let $\eta_i(y), \mu_i(y)$ be the unique solutions of the following cellular problems in Y_3:

$$\textbf{(cell.13}^{\textbf{i}}) \begin{cases} -\mathrm{d}iv_y[\mathbb{A}^3\mathrm{e}^y(\eta_i)(y)] = \frac{1}{3}e_i \text{ in } Y_3, \\ \eta_i(y) = 0 \quad \text{on } Y_{13}, \ \eta_i(y) = 0 \quad \text{on } Y_{23}, \\ y \mapsto \eta_i, [\mathbb{A}^3\mathrm{e}^y(\eta_i)(y)]n_3(y)_{|\partial Y_3 \cap \partial Y}\, Y\text{—periodic,} \end{cases}$$

Homogenization results

$$
(\textbf{cell.23}^{\mathbf{i}}) \begin{cases} -\mathrm{d}iv_y[\mathbb{A}^3 \mathrm{e}^y(\mu_i)(y)] = 0 \text{ in } Y_3, \\ \mu_i(y) = \frac{1}{3} e_i \quad \text{on } Y_{13},\ \mu_i(y) = 0 \quad \text{on } Y_{23}, \\ y \mapsto \mu_i, [\mathbb{A}^3 \mathrm{e}^y(\mu_i)(y)] n_3(y)_{|\partial Y_3 \cap \partial Y}\ Y\text{—periodic} \end{cases}
$$

Homogenization results

Theorem 1

The functions

$$(u_2, v_2, w_3) \in \left(H^1_L(\Omega)\right)^3 \times \left(L^2(\Omega; H^1(Y_\alpha)/\mathbb{R}^3)\right)^3 \times \left(L^2(\Omega; H^1_\sharp(Y_3))\right)^3$$

are the unique solutions of the following homogenized coupled problems : :

$$\begin{cases} -\mathrm{d}iv_x[\mathbb{A}^{2,hom}\mathrm{e}^x(u_2)(x)] + \int_{Y_{23}} \mathbb{A}^3\mathrm{e}^y(w_3)(x,y)n_3(y)dS(y) = \theta_2 F \text{ in } \Omega, \\ v_2(x,y) = \mathrm{e}^x_{kh}(u_2)(x)w_2^{kh}(y), \end{cases}$$

where w_2^{kh} and $\mathbb{A}^{2,\text{hom}}$ are defined above.

Homogenization results

Theorem 1

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are the unique solutions of the following homogenized coupled problems : :

$$(P_1) \begin{cases} -\mathrm{d}iv_{\widetilde{y}}[\widetilde{A^1}\widetilde{\mathrm{e}}^{\widetilde{y}}(v_1)(x,\widetilde{y})] - \dfrac{\partial}{\partial x_3}[A^1_{33}\mathrm{e}^x_3(v_1)(x,\widetilde{y})] = F \quad \text{in } \widetilde{Y}_1, \\ \widetilde{A^1}\widetilde{\mathrm{e}}^{\widetilde{y}}(v_1)n_3 = \left\langle \mathbb{A}^3 \mathrm{e}^y(w_3)(x,y)n_3 \right\rangle_{\mathrm{I}} \quad \text{on } \widetilde{Y}_{13}, \end{cases}$$

where $\langle . \rangle_{\mathrm{I}}$ denotes the integration with respect to y_3 over I.

Homogenization results

Theorem 1

The functions

$$(u_2, v_2, w_3) \in \left(H^1_L(\Omega)\right)^3 \times \left(L^2(\Omega; H^1(Y_\alpha)/\mathbb{R}^3)\right)^3 \times \left(L^2(\Omega; H^1_\sharp(Y_3))\right)^3$$

are the unique solutions of the following homogenized coupled problems : :

$$(P_3) \begin{cases} -\mathrm{d}iv_y \left[\mathbb{A}^3 \mathrm{e}^y(w_3)(x,y)\right] = F \text{ in } Y_3, \\ w_3(x,y) = v_1(x,\widetilde{y}) - u_2(x) \text{ on } Y_{13}, \\ w_3(x,y) = 0 \text{ on } Y_{23}, \\ y \mapsto w_3(x,y) \; Y-periodic. \end{cases}$$

Remark

The problems (P_1) and (P_3) involve both macroscopic and microscopic variables (x, y), a unique macroscopic function u_2 and two microscopic functions v_1, w_3. These functions are "strongly" coupled via the following boundary conditions :

$$\widetilde{A^1}\widetilde{e}^{\widetilde{y}}(v_1)n_3 = \left\langle \mathbb{A}^3 e^y(w_3)(x,y)n_3 \right\rangle_{\mathrm{I}} \text{ on } \widetilde{Y}_{13}$$

with

$$w_3(x,y) = v_1(x,\widetilde{y}) - u_2(x) \text{ on } Y_{13}.$$

As a consequence, the above interface conditions display a remarkable displacement jump and traction continuity. Thus, the effects of a very compliant interphase shown by *Hashin [2002]* are recovered.

Corrector results

Let v_1, u_2, v_2, w_3 be as in Theorem 1. We define the following sequences of functions

$$
\begin{aligned}
\xi_1(x,y) &:= \chi_1(y)E^{\widetilde{y},x}(v_1)(x,\widetilde{y}),\\
\xi_2(x,y) &:= \chi_2(y)\left(\mathrm{e}^x(u_2)(x)+\mathrm{e}^y v_2(x,y)\right),\\
\xi_3(x,y) &:= \chi_3(y)\mathrm{e}^y(w_3)(x,y),\\
\xi_k^\varepsilon(x) &:= \chi_k^\varepsilon(x)\xi_k^\varepsilon(x,\frac{x}{\varepsilon}),\ \ k=1,2,3,\\
\mathbb{B}_1^\varepsilon(x) &:= \chi_1^\varepsilon(x)\left(\begin{array}{c|c}\varepsilon\widetilde{A}^1 & \varepsilon C^1\\ \hline \varepsilon L^1 & A_{33}^1\end{array}\right),\\
\mathbb{B}_2^\varepsilon(x) &:= \chi_2^\varepsilon(x)\mathbb{A}_2,\ \ \mathbb{B}_3^\varepsilon(x) := \varepsilon\chi_3^\varepsilon(x)\mathbb{A}_3,
\end{aligned}
$$

where $E^{\widetilde{y},x} = \begin{pmatrix}\mathrm{e}^{\widetilde{y}}_{\alpha\beta}\\ \mathrm{e}^x_3\end{pmatrix}$.

Homogenization results

Theorem 2
If the functions, $\mathrm{e}^{\tilde{y}}_{\alpha\beta}(v_1)$, $\mathrm{e}^{y}(v_2)$ and $\mathrm{e}^{y}(w_3)$ are admissible then

$$\limsup_{\varepsilon\searrow 0} \|\chi_1^\varepsilon \left(E_\varepsilon^x(u^\varepsilon) - \xi_1^\varepsilon\right)\|_{L^2(\Omega)} = 0,$$

$$\limsup_{\varepsilon\searrow 0} \|\chi_2^\varepsilon \left(\mathrm{e}^x(u^\varepsilon) - \xi_2^\varepsilon\right)\|_{L^2(\Omega)} = 0,$$

$$\limsup_{\varepsilon\searrow 0} \|\chi_3^\varepsilon \left(\varepsilon\mathrm{e}^x(u^\varepsilon) - \xi_3^\varepsilon\right)\|_{L^2(\Omega)} = 0.$$

Where E_ε^x is defined by $E_\varepsilon^x := \begin{pmatrix} \varepsilon \mathrm{e}^{\tilde{x}}_{\alpha\beta} \\ \mathrm{e}^x_3 \end{pmatrix}.$

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Thank you for your attention.