Global exact controllability of the bilinear Schrödinger equation on compact graphs

Alessandro Duca

Workshop on PDE's

Modelling & Theory,

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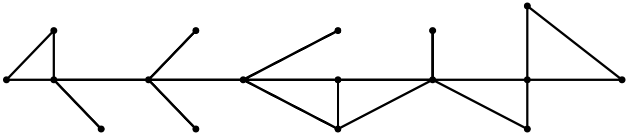
Introduction



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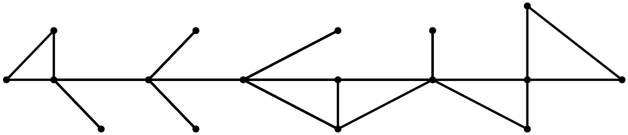

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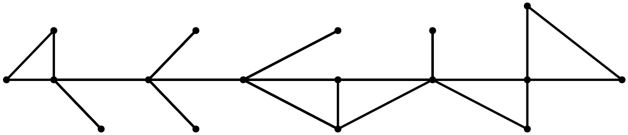

Compact metric graph

- Set of points (vertices) connected by segments (edges),
- equipped with a metric structure (equipped with a distance),
- composed by a finite number of vertices and edges of finite length.

Let $\mathscr{G}$ be a compact metric graph. In $\mathscr{H} = L^2(\mathscr{G}, \mathbb{C})$ equipped with the norm $\|\cdot\|$, we consider the bilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi(t) = -\Delta\psi(t) + u(t)B\psi(t), & t \in (0, T), \\ \psi(0) = \psi^0 & T > 0. \end{cases} \tag{BSE}$$

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Aim: Study the controllability of the (BSE) in a suitable $\mathcal{M} \subset \mathscr{H}$ according to the boundary conditions and structure of $\mathscr{G}$.

$$\forall \psi^1, \psi^2 \in \mathcal{M} \ : \|\psi^1\| = \|\psi^2\|, \ \exists T, \ u \implies \Gamma_T^u \psi^1 = \psi^2.$$

Boundary conditions for $D(-\Delta)$

- **Neumann-Kirchhoff ($\mathcal{NK}$)** in v (internal vertex)

$$(\mathcal{NK}): \quad \begin{cases} f \text{ is continuous in } v, \\ \sum_{e \in N(v)} \sigma_e(v) \frac{\partial f}{\partial x_e}(v) = 0, \end{cases}$$

where $N(v)$ is the set of edges containing v and

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- **Dirichlet ($\mathcal{D}$)** or **Neumann boundary conditions ($\mathcal{N}$)** in the external vertices.

$$(\mathcal{D}): \; f(v) = 0, \qquad (\mathcal{N}): \; \frac{\partial f}{\partial x}(v) = 0.$$

Some Literature for $\mathscr{G} = [a, b]$ and $a < b$

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Peculiarities of the problem

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the (ordered) spectrum of $-\Delta$.

$$\text{If } \mathscr{G} = [a, b] \text{ for } a < b \quad \Longrightarrow \quad \inf_{k\in\mathbb{N}} |\lambda_{k+1} - \lambda_k| > 0, \qquad (1)$$

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(N is the number of edges of $\mathscr{G}$). We need more. Let $d \geq 0$.

Assumptions A.1(d): There exists $C>0$ so that

$$|\lambda_{k+1} - \lambda_k| \geq \frac{C}{k^d}, \qquad \forall k \in \mathbb{N}.$$

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Assumptions A.2(η, d):

- We have $B : H^2_{\mathscr{G}} \to H^2_{\mathscr{G}}$ and

$$B : H^{2+\eta+d}_{\mathscr{G}} \to H^{2+\eta+d} \cap H^2_{\mathscr{G}}.$$

- There exists $C > 0$ such that

$$|\langle \phi_j, B\phi_1 \rangle| \geq \frac{C}{j^{2+\eta}}, \qquad \forall j \in \mathbb{N}.$$

- For every $j, k, l, m \in \mathbb{N}$ such that $\lambda_j - \lambda_k - \lambda_l + \lambda_m = 0$,

$$\langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0.$$

Well-posedness

Theorem (D.)

Let the couple $(-\Delta, B)$ verify Assumptions A.1(d) and Assumptions A.2(η, d) with $d + \eta \in [1, 3/2)$. The well-posedness of the (BSE) is guaranteed in $H_{\mathscr{G}}^{2+\eta+d}$.

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The following interpolation proposition is crucial for the well-posedness of the (BSE).

Proposition (D.)

If $\mathscr{G}$ is a graph equipped with Dirichlet and Neumann type boundary conditions, then

$$H_{\mathscr{G}}^{3+s_1} = H_{\mathscr{G}}^3 \cap H^{3+s_1} \quad \textit{for} \quad s_1 \in [0, 1/2).$$

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If $(-\Delta, B)$ satisfies Assumptions A.2(η, d) with $d + \eta \in [1, 3/2)$, then the (BSE) is globally exactly controllable in $H_{\mathscr{G}}^{2+d+\eta}$, i.e.

$$\forall \psi^1, \psi^2 \in H_{\mathscr{G}}^{2+\eta+d} : \ \|\psi^1\|_{\mathscr{H}} = \|\psi^2\|_{\mathscr{H}}, \ \exists T > 0, \ u \in L^2((0, T), \mathbb{R}) \\ \Rightarrow \ \Gamma_T^u \psi^1 = \psi^2.$$

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Remark

Under suitable assumptions, the well-posedness and the global exact controllability can also be guaranteed when $\eta + d \in (0, 7/2)$ in $H^{2+\epsilon}_{\mathscr{G}}$ with $\epsilon \in [\max\{\eta + d, 1\}, 7/2)$.

Examples: global exact controllability

Let $B|_{L^2(e_1)} = (x - L_1)^4$ and $B|_{L^2(e_k)} = 0$ with $k \neq 1$.

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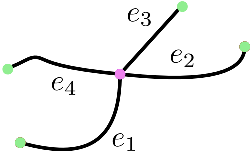

For almost every $\{L_j\}_{j\leq 4}$ such that $\{1, \{L_j\}_{j\leq 4}\}$ are Q-linearly independent and $\forall L_j/L_k$ algebraic irrational numbers,

Neumann-Kirchhoff Dirichlet

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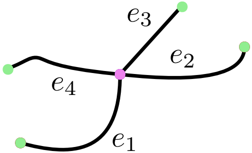

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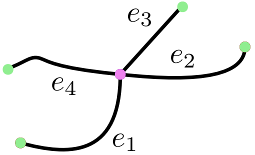

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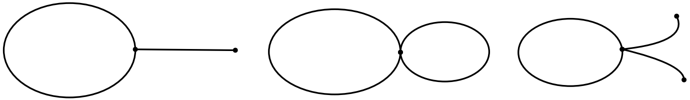
Other examples:



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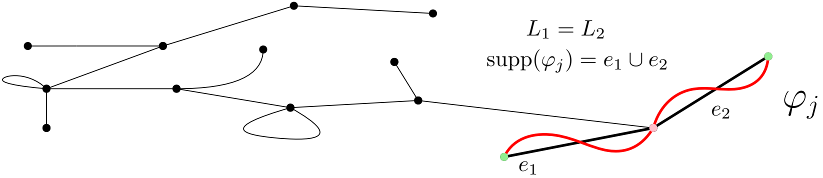

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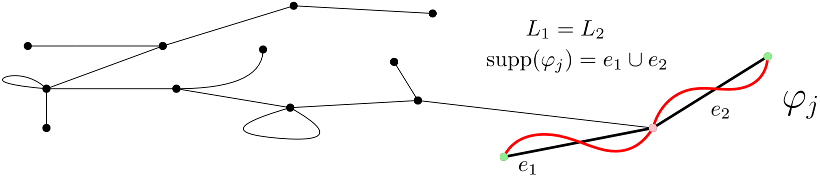

Let $\varphi_j|_{e_1} = -\varphi_j|_{e_2}$ be so that $\varphi_j|_{e_1}$ is the j-th eigenfunction of

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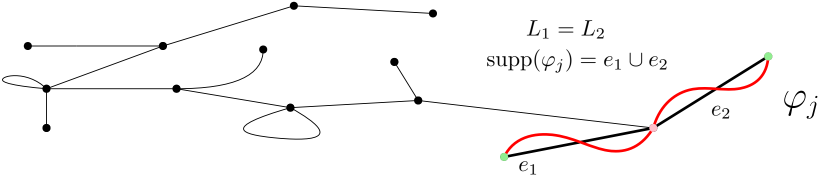

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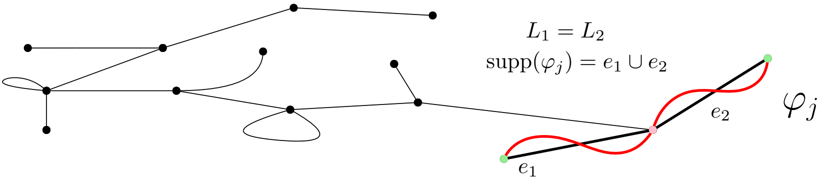

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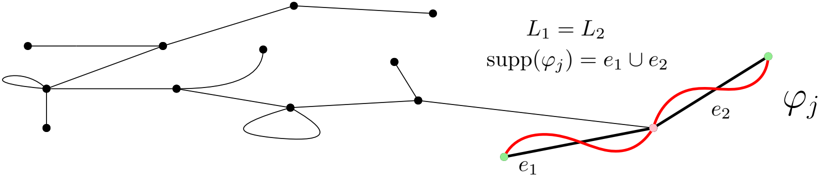

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The spectrum of $-\Delta$ in $\widetilde{\mathscr{H}}$ is explicit and it is possible to verify the validity of Assumptions A.1.

- If $(-\Delta, B)$ satisfies Assumptions A.1(d) and Assumptions A.2(d, η) in $\widetilde{\mathscr{H}}$ for suitable $\eta > 0$ and $d \geq 0$, then the global exact controllability can be guaranted in

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- As $\{\varphi_j\}_{j\in\mathbb{N}} \subset H_{\mathscr{G}}^s$ for every $s > 0$, we have

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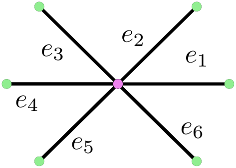
- Let $\{\widetilde{\lambda}_k\}_{k\in\mathbb{N}}$ be the spectrum of $-\Delta$ in $\widetilde{\mathscr{H}}$.

$$\implies \begin{cases} (BSE) \text{ is energetically controllable} \\ \text{with respect to } \{\widetilde{\lambda}_k\}_{k\in\mathbb{N}}. \end{cases}$$

Examples: energetic controllability



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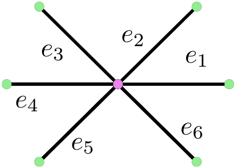

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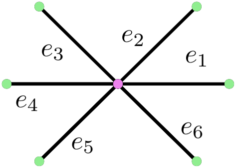

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$$\text{If} \quad L_k = L_j, \qquad \forall k, j \leq 6,$$

$\Longrightarrow$ (BSE) is **energetically controllable** with respect to

$$\left\{ \frac{k^2 \pi^2}{L_1^2} \right\}_{k \in \mathbb{N}}.$$

Introduction

Well-posedness and global exact controllability

Energetic controllability

Peculiarity of the proof

Local exact controllability

The local controllability in a neighborhood of ϕ_1 in $H^s_{\mathscr{G}}$ with $s \geq 0$ corresponds to local surjectivity of the map

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$$\alpha_{k,1} = \langle \phi_k, \Gamma^u_T \phi_1 \rangle, \qquad k \in \mathbb{N}.$$

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$$\alpha_{k,1} = \langle \phi_k, \Gamma^u_T \phi_1 \rangle, \qquad k \in \mathbb{N}.$$

Generalized Inverse Function Theorem $\Rightarrow$ surjectivity of $\gamma := (d_u \alpha(u=0)) \cdot \; v$

$$\gamma_{k,1}(v) = -i \int_0^T v(s) e^{-i(\lambda_1 - \lambda_k)s} ds \langle \phi_k, B\phi_1 \rangle. \qquad (2)$$

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- **New approach:** Interpolation features $\Rightarrow$ well-posedness in $H^{2+d+\eta}_{\mathscr{G}}$ $\Rightarrow$ moment problem in $X(d,\eta)$.

Thank you for your attention!