Imaging small polarizable scatterers with polarization data

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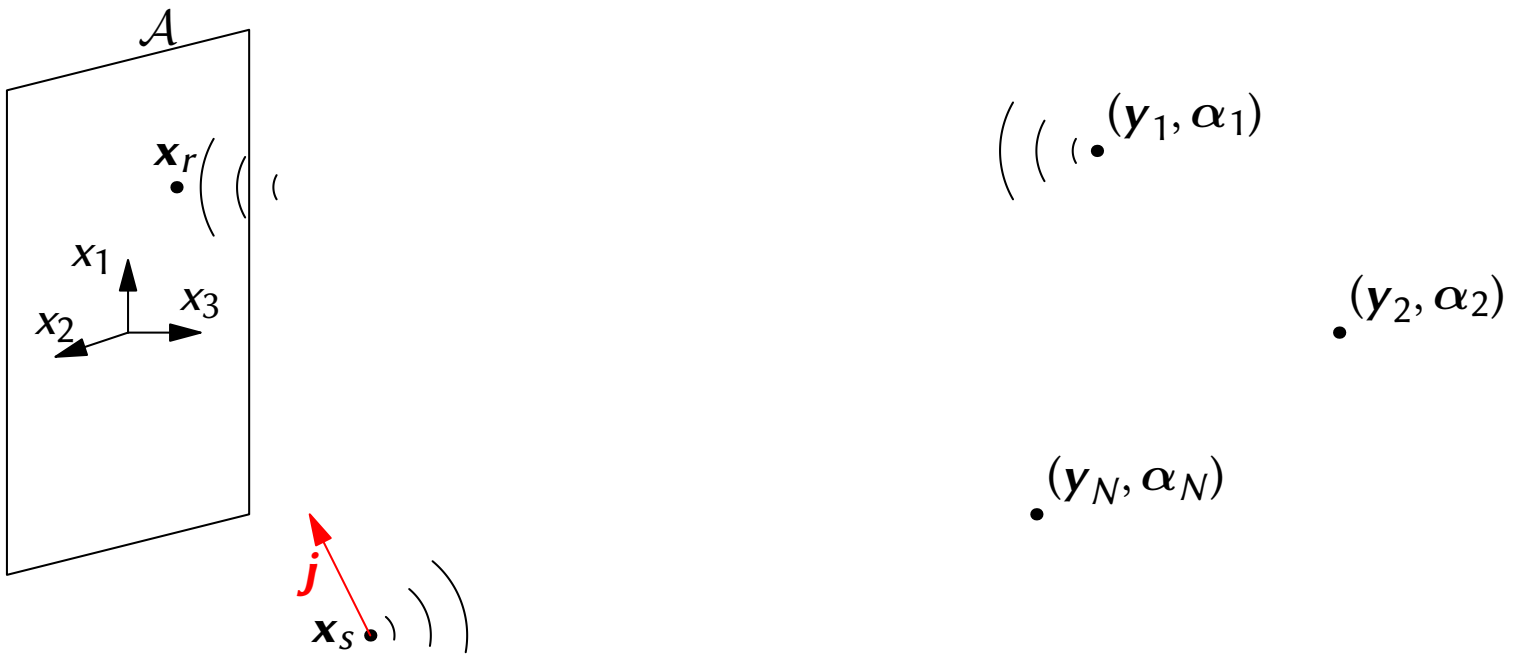
Full data problem

Fact: Small dielectric scatterers can be described by a 3×3 complex symmetric matrix, the polarization tensor (e.g. Vogelius, Volkov 2000).

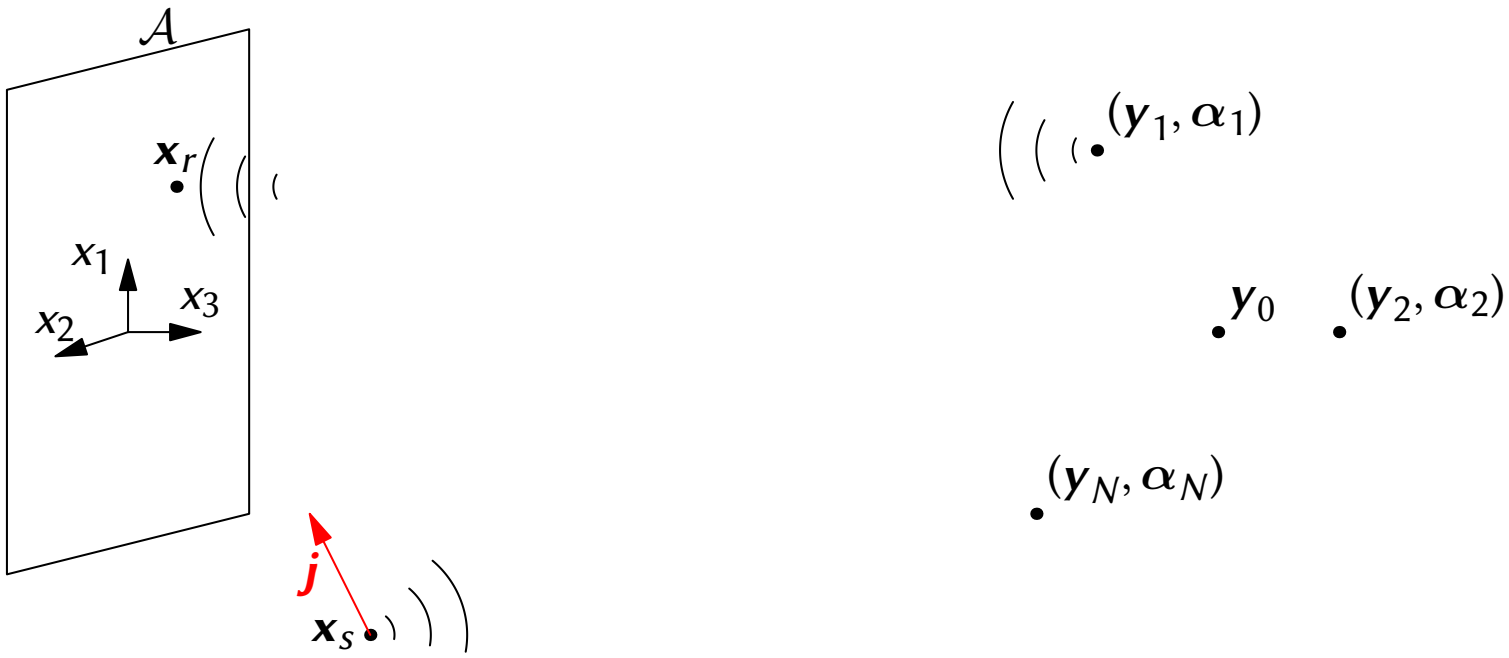


Problem:

Find positions $\boldsymbol{y}_j$ and polarization tensors $\boldsymbol{\alpha}_j$ from $\boldsymbol{\Pi}(\boldsymbol{x}_r, k)$, for $\boldsymbol{x}_r \in \mathcal{A}$.

Here $\boldsymbol{\Pi}(\boldsymbol{x}_r, k)\boldsymbol{j} \equiv$ scattered electric field measured at $\boldsymbol{x}_r$ resulting from source with (electric) dipole moment $\boldsymbol{j} \in \mathbb{C}^3$.

Coherency matrix data problem



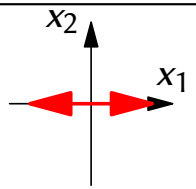
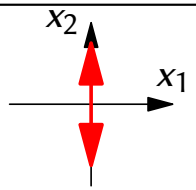
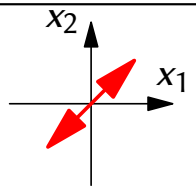
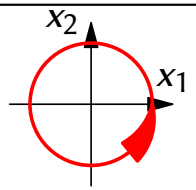
Problem

Find positions $\boldsymbol{y}_j$ and polarization tensors $\boldsymbol{\alpha}_j$ from coherency matrix at $\boldsymbol{x}_r \in \mathcal{A}$

$$\psi(\boldsymbol{x}_r, k) := \left\langle \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^* \right\rangle .$$

Here $\langle \boldsymbol{j} \rangle = \boldsymbol{0}$, $\langle \boldsymbol{j}\boldsymbol{j}^* \rangle = \boldsymbol{U}_s \boldsymbol{U}_s^T$, where $\mathcal{R}(\boldsymbol{U}_s) = (\boldsymbol{x}_s - \boldsymbol{y}_0)^\perp$ and $\boldsymbol{y}_0 \equiv$ ref. point.

Polarization of electromagnetic waves

	coherency matrix	Stokes parameters
	$\left\langle \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^* \right\rangle$	$I = \left\langle \lvert E_1 \rvert^2 + \lvert E_2 \rvert^2 \right\rangle$ $Q = \left\langle \lvert E_1 \rvert^2 - \lvert E_2 \rvert^2 \right\rangle$ $U = \left\langle 2\mathrm{Re}\left(E_1 \overline{E_2}\right) \right\rangle$ $V = \left\langle 2\mathrm{Im}\left(E_1 \overline{E_2}\right) \right\rangle$
	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$[1, 1, 0, 0]$
	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$[1, -1, 0, 0]$
	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$[1, 0, 1, 0]$
	$\frac{1}{2}\begin{bmatrix} 1 & \imath \\ -\imath & 1 \end{bmatrix}$	$[1, 0, 0, 1]$

$\rightsquigarrow$ can be measured with linear polarizer + quarter wave plate

Roadmap

1. Under single scattering (Born) approximation and with $\widetilde{\mathbf{\Pi}} := \boldsymbol{U}_{\|}^{T}\mathbf{\Pi}\boldsymbol{U}_{s}$,

$$\underbrace{\boldsymbol{\psi}}_{4\times\text{real}} = \boldsymbol{\psi}_{\text{inc}} + \underbrace{\widetilde{\mathbf{\Pi}}}_{8\times\text{real}}\boldsymbol{A} + \boldsymbol{A}^{*}\underbrace{\widetilde{\mathbf{\Pi}}^{*}}_{8\times\text{real}} + \widetilde{\mathbf{\Pi}}\boldsymbol{C}\widetilde{\mathbf{\Pi}}^{*},$$

   where $\boldsymbol{U}_{\|} := [\boldsymbol{e}_1, \boldsymbol{e}_2]$.

2. Preprocess data:

$$p(\boldsymbol{\psi}) := \boldsymbol{U}_{\|}(\boldsymbol{\psi} - \boldsymbol{\psi}_{inc})\boldsymbol{A}^{-1}\boldsymbol{U}_{s}^{*} = \mathbf{\Pi} + \text{ error.}$$

3. Under mild conditions error does not matter when imaging:

$$\mathcal{I}_{\text{KM}}[p(\boldsymbol{\psi})] \approx \mathcal{I}_{\text{KM}}[\mathbf{\Pi}],$$

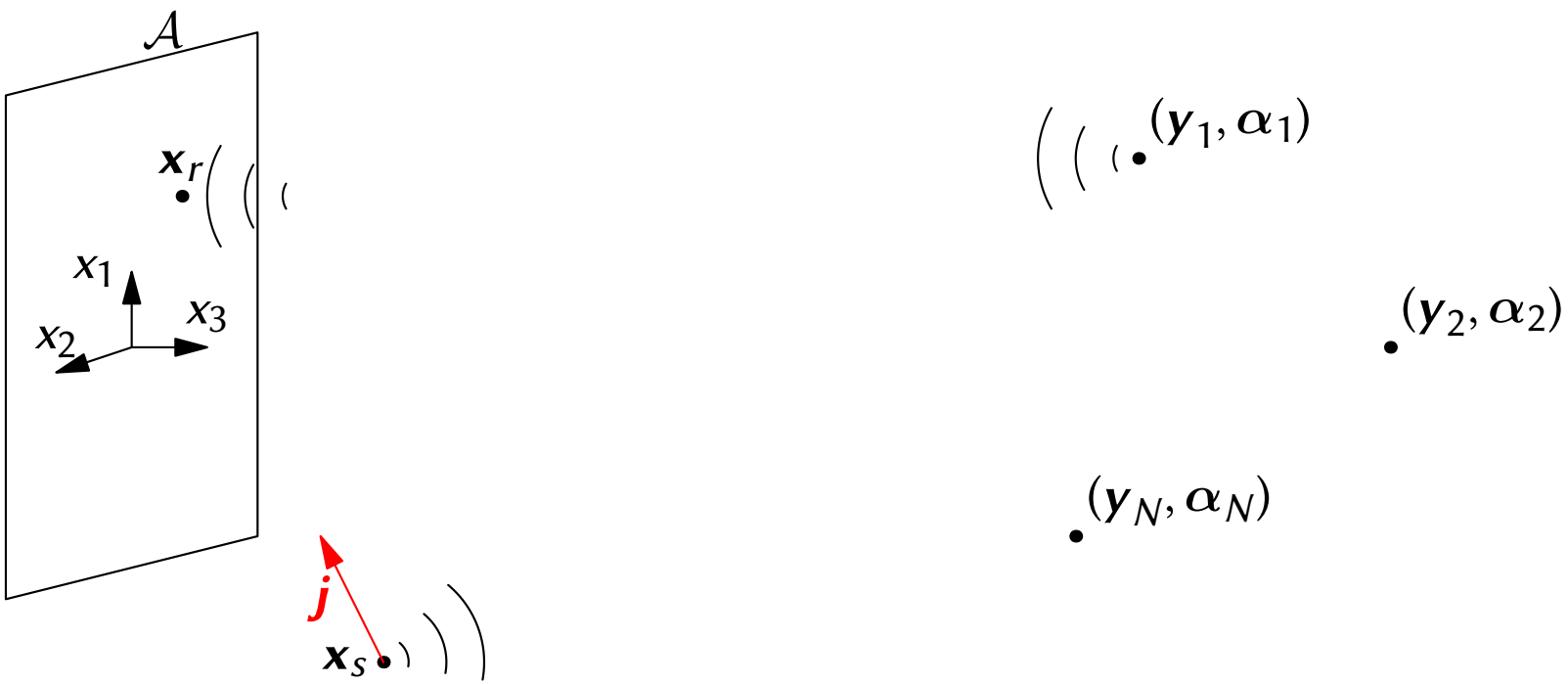
   where $\mathcal{I}_{\text{KM}}[\cdot]$ is a (matrix valued) electromagnetic version of Kirchhoff migration.

4. Extract locations and polarization tensor from $\mathcal{I}_{\text{KM}}[p(\boldsymbol{\psi})]$.

Part 1. Electromagnetic Kirchhoff migration

Part 2. Retrieving full data from polarization data

Part 1. Electromagnetic Kirchhoff migration



Related work

- Integral equation method (Ammari, Moskow, Vogelius 2003)
- DORT for EM (Antoine, Pinçon, Ramdani, Thierry 2008)
- Random matrix theory + MUSIC (Borcea, Garnier 2016)
- Far field data (Chen, Huang 2015, 2016)
- Kirchhoff migration for electromagnetics (Cassier, GV 2017)
- Polarimetric SAR
- Polarization Sensitive OCT (Elbau, Mindrinos, Scherzer 2017)

The model: Born approximation

Ignoring multiple scattering, the total field $\boldsymbol{E}$ for a source with dipole moment $\boldsymbol{j}$ is

$$\boxed{\boldsymbol{E}(\boldsymbol{x},k) = \underbrace{\boldsymbol{G}(\boldsymbol{x},\boldsymbol{x}_s;k)\boldsymbol{j}}_{=\boldsymbol{E}_{\text{inc}}} + \underbrace{\sum_{n=1}^{N} \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y}_n;k)\alpha_n \boldsymbol{G}(\boldsymbol{y}_n,\boldsymbol{x}_s;k)\boldsymbol{j}}_{=\boldsymbol{E}_{\text{scatt}}},}$$

where $\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y};k)$ is the 3×3 dyadic Green function:

$$\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y};k) = G(\boldsymbol{x},\boldsymbol{y};k)\left[(1+m(kr))\boldsymbol{I} - (1+3m(kr))\frac{\boldsymbol{r}\boldsymbol{r}^\top}{r^2}\right],$$

with $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}$, $r = |\boldsymbol{r}|$ and the scalar Green function is

$$G(\boldsymbol{x},\boldsymbol{y};k) = \frac{\exp[\imath kr]}{4\pi r} \text{ and } m(kr) = \frac{\imath kr - 1}{(kr)^2}.$$

The data and the imaging function

- The full data is scattered field at $\boldsymbol{x}_r \in \mathcal{A}$ for arbitrary source dipole moment, i.e.

$$\boldsymbol{\Pi}(\boldsymbol{x}_r, k) := \sum_{n=1}^{N} \boldsymbol{G}(\boldsymbol{x}_r, \boldsymbol{y}_n; k)\boldsymbol{\alpha}_n \boldsymbol{G}(\boldsymbol{y}_n, \boldsymbol{x}_s; k) \in \mathbb{C}^{3\times 3}.$$

- The Kirchhoff imaging function is the matrix field:

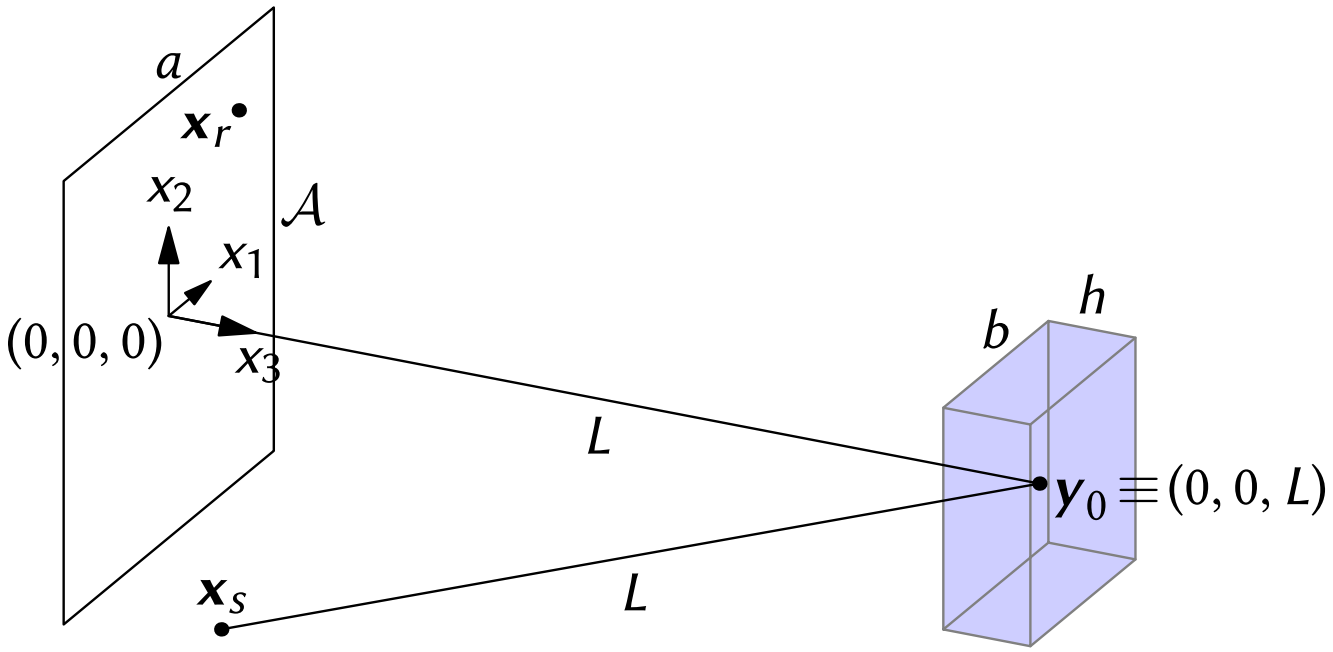
$$\mathcal{I}_{\mathrm{KM}}[\boldsymbol{\Pi}](\boldsymbol{y}) = \int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \overline{\boldsymbol{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)} \boldsymbol{\Pi}(\boldsymbol{x}_r; k) \overline{\boldsymbol{G}(\boldsymbol{y}, \boldsymbol{x}_s; k)} \in \mathbb{C}^{3\times 3}.$$

In acoustics, Kirchhoff imaging is well-understood:

- image peaks at scatterer locations
- Resolution estimates: $\lambda L/a$ in cross-range (Rayleigh criterion), c/B in range (see e.g., Bleistein, Cohen and Stockwell 2001)

$\rightsquigarrow$ Similar results hold in electromagnetics.

Fraunhofer regime: paraxial approx. of Maxwell equations



- Assumption on the characteristic lengths: $kL \gg 1$, $a, b, h \ll L$,
- Fresnel numbers:

$$1 \ll \Theta_a = \frac{ka^2}{L} = \frac{2\pi a}{\lambda L/a} \ll \frac{L^2}{a^2} \text{ and } \Theta_b = \frac{kb^2}{L} \ll 1$$

- Depth: $kh = \mathcal{O}(1)$
- Source: $|\boldsymbol{x}_s - \boldsymbol{y}_0| = L$ (can be relaxed)

Dyadic Green function in Fraunhofer regime

- The dyadic Green function is

$$\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y};k) = G(\boldsymbol{x},\boldsymbol{y};k)\left[(1+m(kr))\boldsymbol{I} - (1+3m(kr))\frac{\boldsymbol{r}\boldsymbol{r}^\top}{r^2}\right].$$

- Phase approximation with $\boldsymbol{x} = (\boldsymbol{x}_\parallel, 0)$ and $\boldsymbol{y} = (\boldsymbol{y}_\parallel, L+\eta)$

$$k|\boldsymbol{x}-\boldsymbol{y}| = k(L+\eta) + k\frac{|\boldsymbol{x}_\parallel|^2}{2L} + k\frac{\boldsymbol{x}_\parallel\cdot\boldsymbol{y}_\parallel}{L} + o(1).$$

- In Fraunhofer regime: $G(\boldsymbol{x},\boldsymbol{y};k) \approx \mathcal{G}(\boldsymbol{x},\boldsymbol{y};k)$, where

$$\mathcal{G}(\boldsymbol{x},\boldsymbol{y};k) = \frac{1}{4\pi L}\exp\left[\imath\left[k(L+\eta) + \frac{k|\boldsymbol{x}_\parallel|^2}{2L} + \frac{k\boldsymbol{x}_\parallel\cdot\boldsymbol{y}_\parallel}{L}\right]\right].$$

- Since $m(k\boldsymbol{r}) = \mathcal{O}(1/(kL))$, we get

$$\boxed{\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y};k) \approx \mathcal{G}(\boldsymbol{x},\boldsymbol{y};k)\,\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}) \quad \text{with} \quad \boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{I} - \frac{\boldsymbol{r}\boldsymbol{r}^\top}{r^2}.}$$

Kirchhoff imaging function in Fraunhofer regime

$$\mathcal{I}_{\mathrm{KM}}[\boldsymbol{\Pi}](\boldsymbol{y}) = \int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \overline{\boldsymbol{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)} \boldsymbol{\Pi}(\boldsymbol{x}_r; k) \overline{\boldsymbol{G}(\boldsymbol{y}, \boldsymbol{x}_s; k)}$$

$$= \sum_{n=1}^{N} \underbrace{\int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \overline{\boldsymbol{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)} \boldsymbol{G}(\boldsymbol{x}_r, \boldsymbol{y}_n; k)}_{\approx \boldsymbol{H}_r(\boldsymbol{y}, \boldsymbol{y}_n; k)} \alpha_n \underbrace{\boldsymbol{G}(\boldsymbol{y}_n, \boldsymbol{x}_s; k) \overline{\boldsymbol{G}(\boldsymbol{y}, \boldsymbol{x}_s; k)}}_{= \boldsymbol{H}_s(\boldsymbol{y}, \boldsymbol{y}_n; k)^T}$$

where

$$\boldsymbol{H}_r(\boldsymbol{y}, \boldsymbol{y}'; k) = \int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \overline{\mathcal{G}(\boldsymbol{x}_r, \boldsymbol{y}; k)} \mathcal{G}(\boldsymbol{x}_r, \boldsymbol{y}'; k) \boldsymbol{P}(\boldsymbol{x}_r, \boldsymbol{y}) \boldsymbol{P}(\boldsymbol{x}_r, \boldsymbol{y}').$$

With $\boldsymbol{y} = (\boldsymbol{y}_\|, \eta)$ and $\boldsymbol{y}' = (\boldsymbol{y}'_\|, \eta')$:

$$\boxed{\boldsymbol{H}_r(\boldsymbol{y}, \boldsymbol{y}'; k) = \frac{\exp[\imath k(\eta' - \eta)]}{(4\pi L)^2} \int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \exp\left[\imath k \left[\frac{\boldsymbol{x}_{r,\|} \cdot (\boldsymbol{y}'_\| - \boldsymbol{y}_\|)}{L}\right]\right] \boldsymbol{P}(\boldsymbol{x}_r, \boldsymbol{y}) \boldsymbol{P}(\boldsymbol{x}_r, \boldsymbol{y}').}$$

$\rightsquigarrow$ $\boldsymbol{H}_r(\boldsymbol{y}, \boldsymbol{y}'; k)$ is a matrix valued point spread function.

Summary of resolution analysis

- Image decays in range: Norm of KM image of a single scatterer at $\boldsymbol{y}_*$ decays in range as $\mathcal{O}((a/L)^2 L/(ak|(\boldsymbol{y}-\boldsymbol{y}_*)_{\|}|))$.

- Polarization tensor reconstruction: one can recover a 2×2 matrix representing the action of the polarization tensor on left and right subspaces dictated by experimental setup
(i.e. $\perp$ of line of sight receiver – scatterer and source – scatterer)

- Polarization tensor reconstruction error: At dipole location $\boldsymbol{y}_i$, error in polarization tensor estimation is $\mathcal{O}(L/(ak \min_{n\neq i} |(\boldsymbol{y}_n - \boldsymbol{y}_i)_{\|}|))$.

- Polarization tensor reconstruction decay: At $\boldsymbol{y}$ away from all dipoles, the estimation of polarization tensor decays as
$\mathcal{O}(L/(ak \min_{n=1,\ldots,N} |(\boldsymbol{y}-\boldsymbol{y}_n)_{\|}|))$.

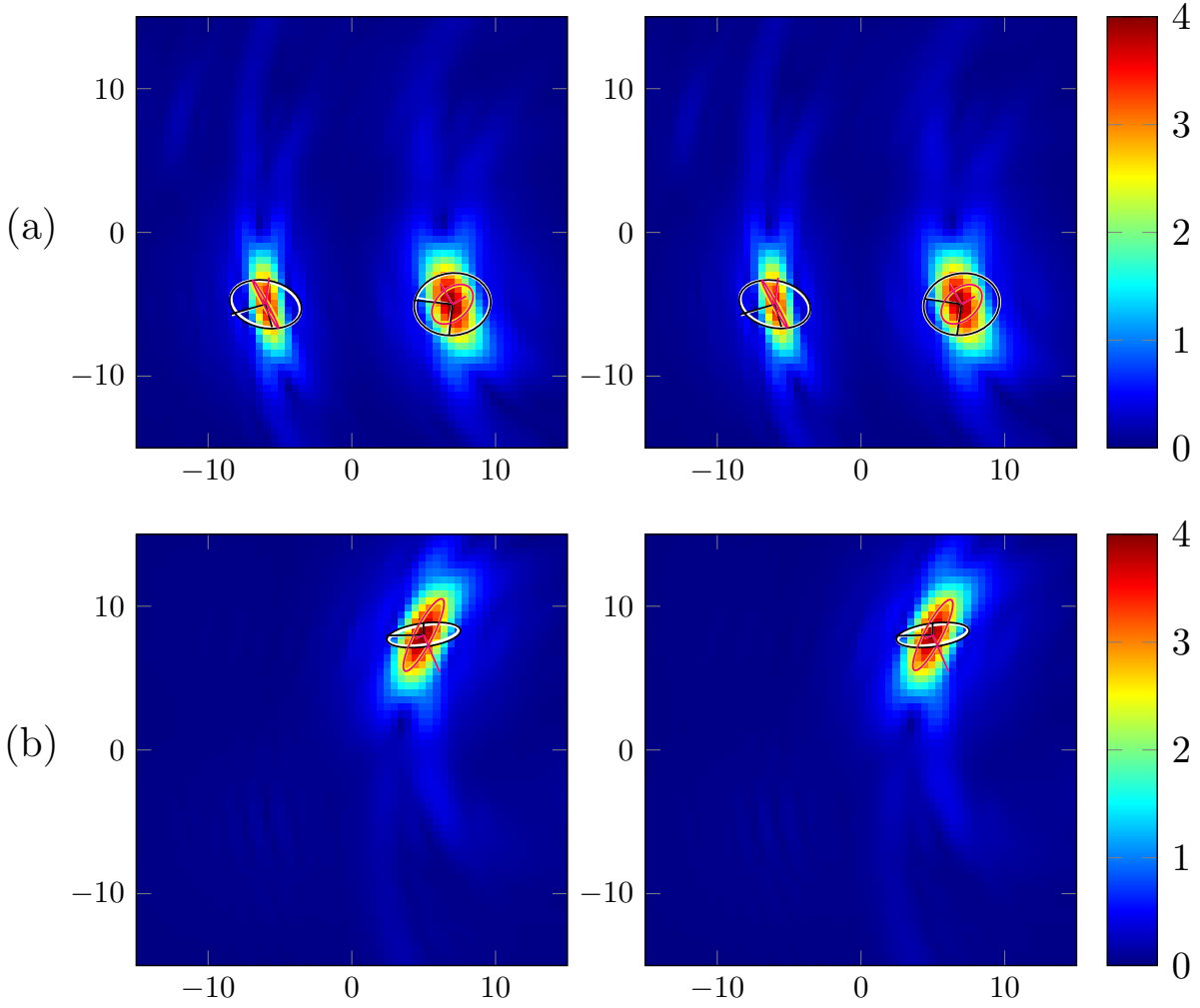
- Range resolution: image of dipole at $(\boldsymbol{y}_*, \eta_*)$ decays as $c/(B|\eta-\eta_*|)$, where B =bandwidth.

Numerical experiments

- $B = [1.2, 3.6]$ GHz ($\lambda_0 = 12.5$cm, $c = 3 \times 10^8 \text{ms}^{-1}$, $\Delta B = B/60$)
- $a = 20\lambda_0$ (sampled at 3 ppw)
- $L = 100\lambda_0$
- Source $\boldsymbol{x}_s = [50, 0, 100(1 - \sqrt{3}/4)]\lambda_0$, s.t. $|\boldsymbol{x}_s - \boldsymbol{y}_0| = 100\lambda_0$.
- Most experiments with three point scatterers given by

	$n = 1$	$n = 2$	$n = 3$
$\boldsymbol{y}_n$	$[-6, -5, 100]\lambda_0$	$[7, -5, 100]\lambda_0$	$[5, 8, 106]\lambda_0$
$\boldsymbol{\alpha}_n$	$\begin{bmatrix} 2+\imath & -\imath & 1 \\ -\imath & 1+2\imath & \imath \\ 1 & \imath & 1+\imath \end{bmatrix}$	$\begin{bmatrix} 2+2\imath & -1+\imath & \imath/2 \\ -1+\imath & 1+2\imath & 0 \\ \imath/2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2-2\imath & 1+\imath & 0 \\ 1+\imath & 1+2\imath & (1-\imath)/2 \\ 0 & (1-\imath)/2 & \imath \end{bmatrix}$

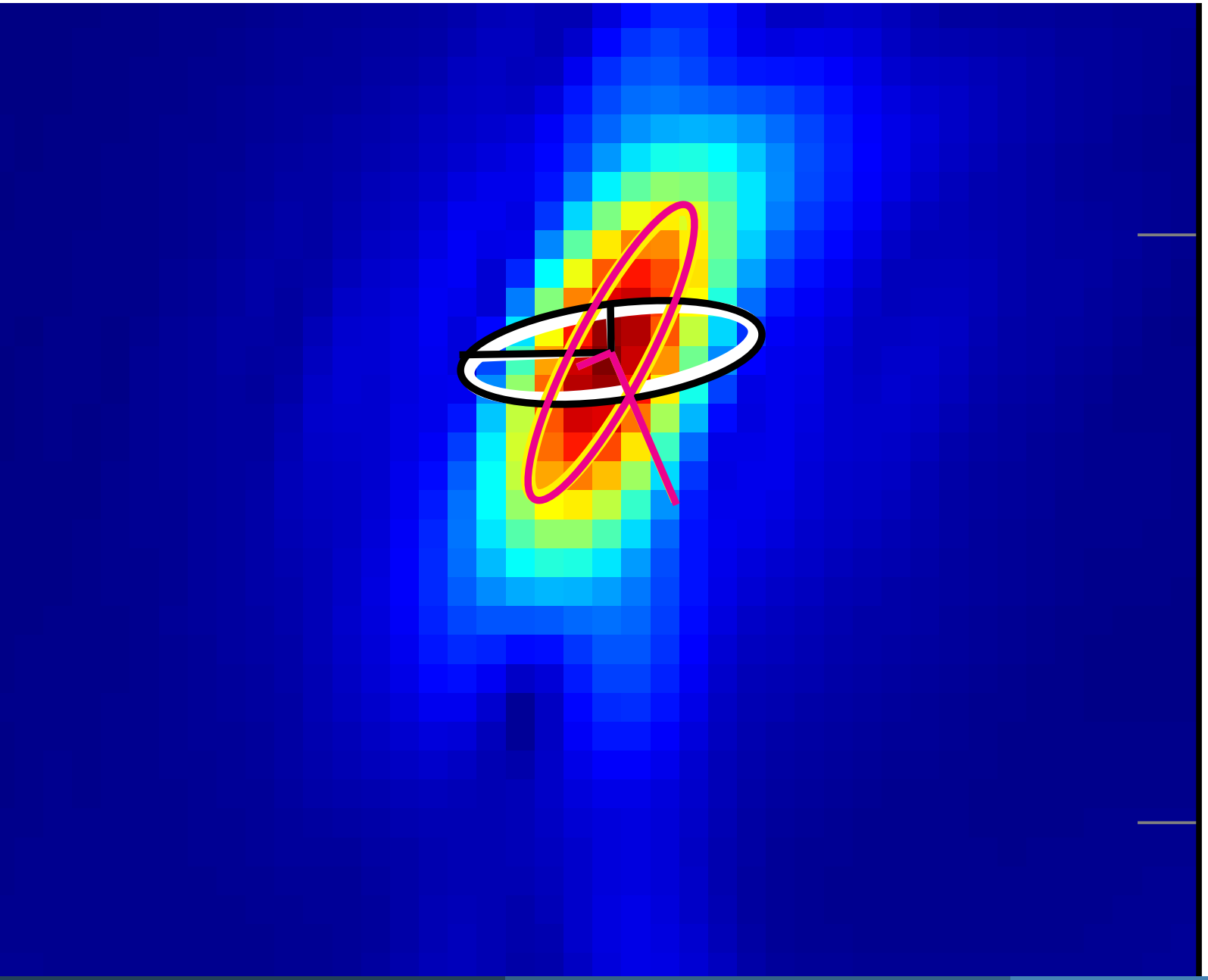
KM image in cross-range



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

(a) $z = 100\lambda_0$, (b) $z = 106\lambda_0$. left: true, right: reconstructed.

KM image in cross-range



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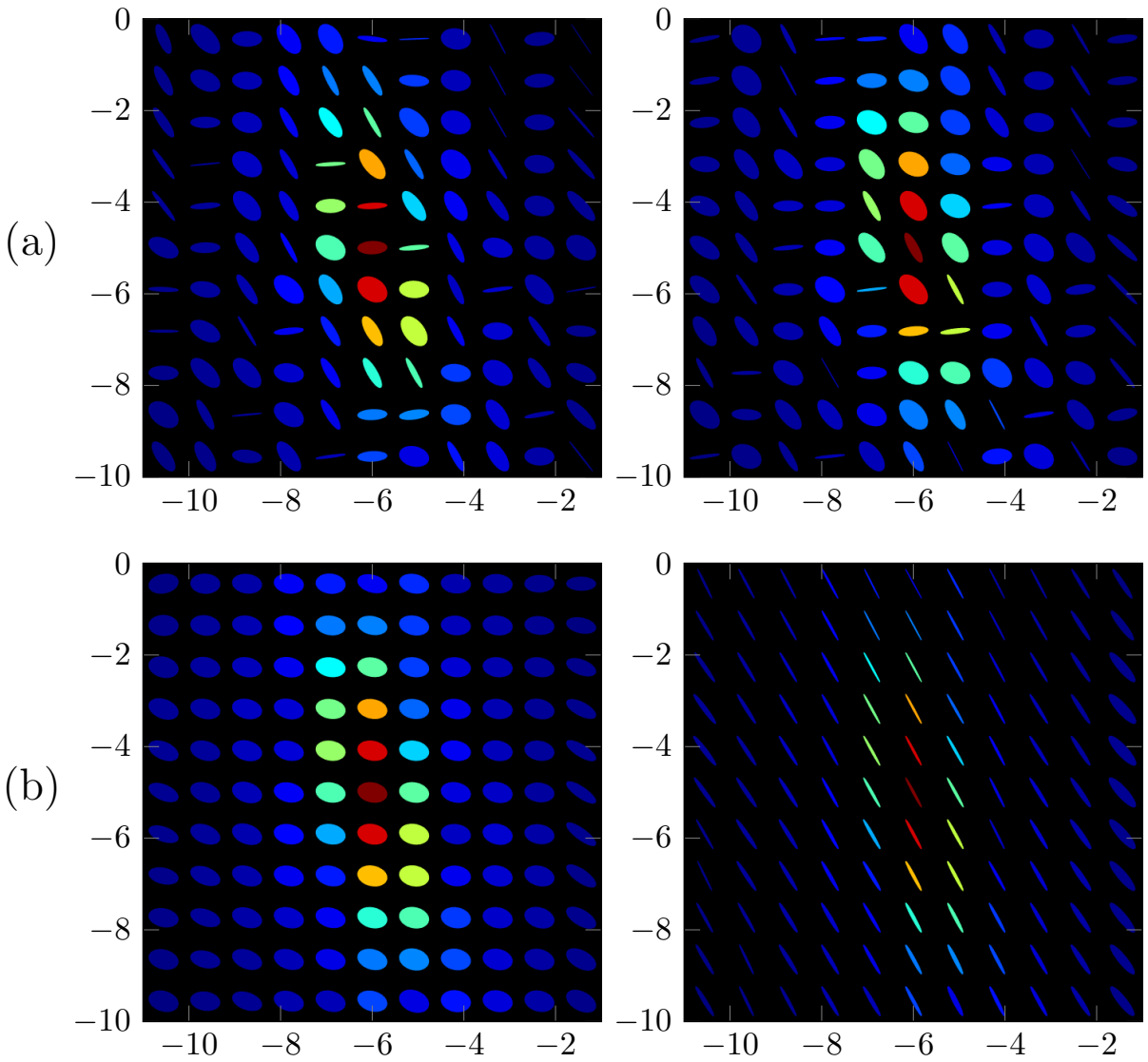
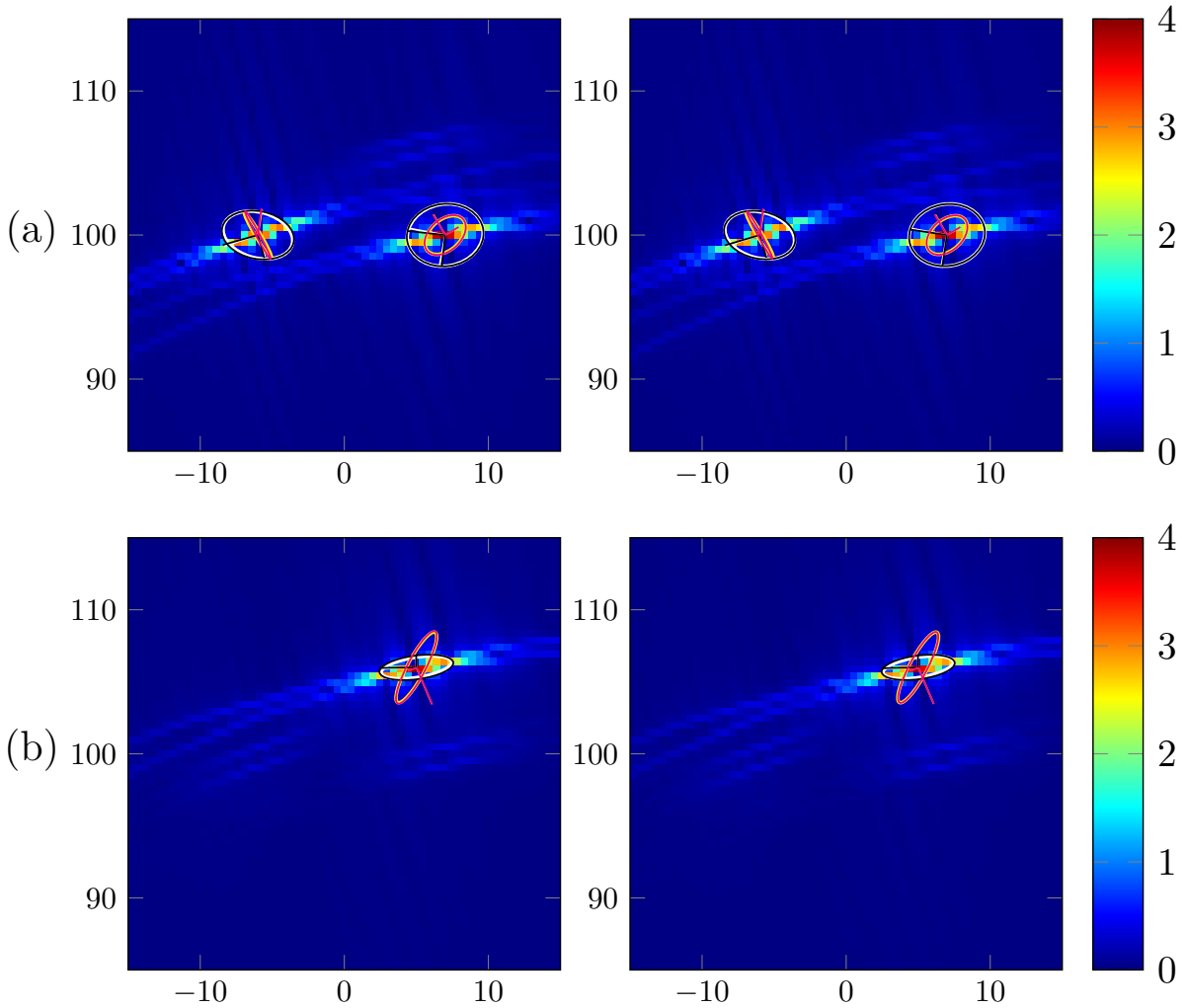
0

KM image in cross-range + stabilization $(\overline{\alpha_{11}}/|\alpha_{11}|)\boldsymbol{\alpha}$



(a): $\boldsymbol{\alpha}$, (b): $(\overline{\alpha_{11}}/|\alpha_{11}|)\boldsymbol{\alpha}$. left: real, right: imaginary.

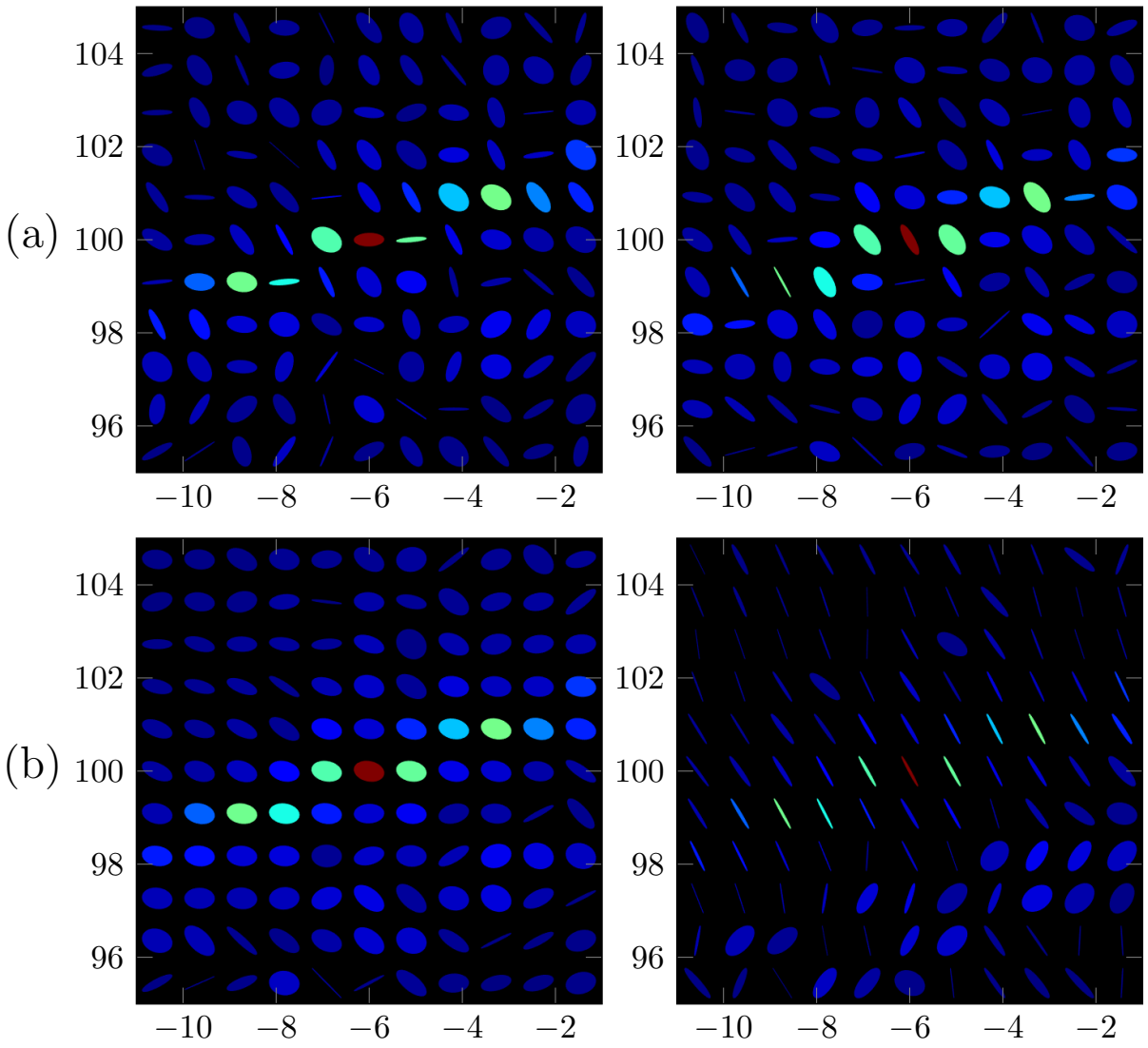
KM image in range



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

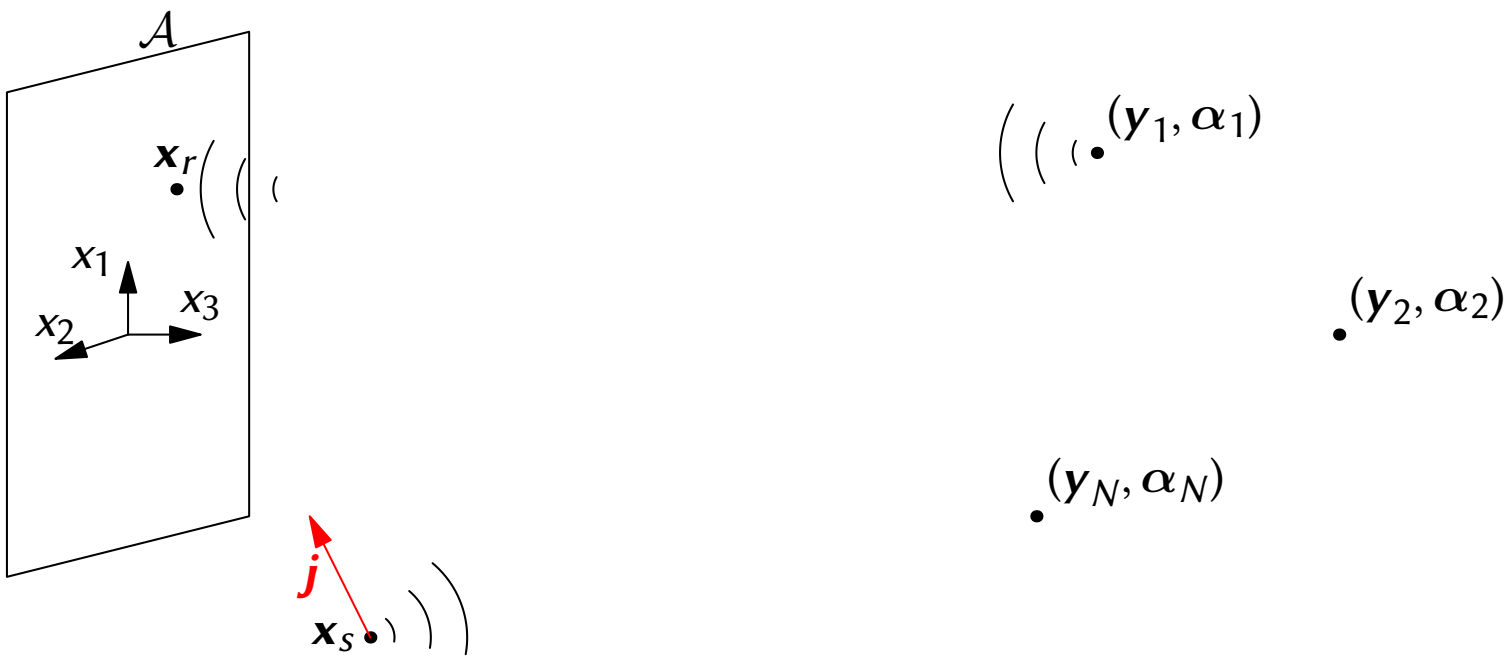
(a) $y = -5\lambda_0$, (b) $y = 8\lambda_0$. left: true, right: reconstructed.

Example of KM image in range + stabilization $(\overline{\alpha_{11}}/|\alpha_{11}|)\boldsymbol{\alpha}$



(a): $\boldsymbol{\alpha}$, (b): $(\overline{\alpha_{11}}/|\alpha_{11}|)\boldsymbol{\alpha}$. left: real, right: imaginary.

Part 2. Retrieving full data from polarization data



Related work: intensity only imaging

Phase retrieval

- Intensity measurements on multiple planes (Gerchberg, Saxton 1972; Gbur, Wolf 2004, . . .)
- Differential identity (Teague 1983, . . .)
- Iterative techniques (Fienup 1987; Crocco, D'Urso, Isernia 2004, . . .)
- Random masks (Fannjiang, Liao 2012) . . .

Bypass phase retrieval

- Model based imaging (Takenaka, Wall, Harada, Tanaka 1997, . . .)
- Convex optimization (Chai, Moscoso, Papanicoloau 2011; Candès, Strohmer, Voroninski 2013; Yin, Xin 2015, . . .)

Partial phase retrieval

- Polarization identity and MUSIC (Novikov, Moscoso, Papanicoloau 2014)
- Kirchhoff migration w/o phases (Bardsley, GV 2015, 2016)
- Far-field asymptotic and Kirchhoff migration (Chen, Huang 2015,2016)

Empirical autocorrelations

- Assumption source dipole moment $\boldsymbol{j}(t)$ is a real stationary Gaussian random process with $\langle \boldsymbol{j}(t) \rangle = \boldsymbol{0}$, $\langle \boldsymbol{j}(t)\boldsymbol{j}(t+\tau)^T \rangle = \boldsymbol{U}_s \widetilde{\boldsymbol{J}}_s(\tau) \boldsymbol{U}_s^T$.
- The empirical autocorrelation of the electric field is

$$\psi_{\text{emp}}(\boldsymbol{x}_r, \tau) = \frac{1}{2T} \int_{-T}^{T} dt\, \boldsymbol{U}_{\|}^* \boldsymbol{E}(\boldsymbol{x}_r, t+\tau) \boldsymbol{E}(\boldsymbol{x}_r, t)^T \boldsymbol{U}_{\|}.$$

Theorem

1. *The expectation $\langle \psi_{emp}(\boldsymbol{x}_r, \tau) \rangle$ is independent of measurement time T.*
2. *The empirical autocorrelation and the coherency matrix are related:*

$$(\langle \psi_{emp}(\boldsymbol{x}_r, \tau) \rangle)^{\wedge}(\omega) = \psi(\boldsymbol{x}_r, \omega).$$

3. *The empirical correlations are ergodic, i.e.*

$$\psi_{emp}(\boldsymbol{x}_r, \tau) \xrightarrow{T \to \infty} \langle \psi_{emp}(\boldsymbol{x}_r, \tau) \rangle .$$

(Similar to acoustics result by Garnier, Papanicolaou 2009)

Relating coherency matrix data and full data

- Assumption: source dipole moment is a Gaussian random vector with

$$\begin{aligned}\langle \boldsymbol{j}(\omega)\rangle &= \mathbf{0},\\ \left\langle \boldsymbol{j}(\omega)\boldsymbol{j}(\omega')^*\right\rangle &= \delta(\omega+\omega')\boldsymbol{U}_s\widetilde{\boldsymbol{J}}_s(\omega)\boldsymbol{U}_s^T,\\ \left\langle \boldsymbol{j}(\omega)\boldsymbol{j}(\omega')^T\right\rangle &= \delta(\omega-\omega')\boldsymbol{U}_s\widetilde{\boldsymbol{J}}_s(\omega)\boldsymbol{U}_s^T.\end{aligned}$$

  $\rightsquigarrow$ for ω fixed $\boldsymbol{j}(\omega)$ is circular Gaussian.

- The coherency matrix data at $\boldsymbol{x}_r \in \mathcal{A}$ and at ω is

$$\begin{aligned}\psi(\boldsymbol{x}_r,\omega) &= \left\langle \boldsymbol{U}_{\|}^T\boldsymbol{E}(\boldsymbol{x}_r,\omega)\boldsymbol{E}^*(\boldsymbol{x}_r,\omega)\boldsymbol{U}_{\|}\right\rangle = \left\langle \begin{bmatrix}E_1\\E_2\end{bmatrix}\begin{bmatrix}E_1\\E_2\end{bmatrix}^*\right\rangle\\ &= \left\langle \boldsymbol{U}_{\|}^T\left[\boldsymbol{G}(\boldsymbol{x}_s,\boldsymbol{x}_r)+\boldsymbol{\Pi}(\boldsymbol{x}_r)\right]\boldsymbol{j}\boldsymbol{j}^*\left[\boldsymbol{G}(\boldsymbol{x}_s,\boldsymbol{x}_r)+\boldsymbol{\Pi}(\boldsymbol{x}_r)\right]^*\boldsymbol{U}_{\|}\right\rangle\\ &= \boldsymbol{U}_{\|}^T\left[\boldsymbol{G}(\boldsymbol{x}_s,\boldsymbol{x}_r)+\boldsymbol{\Pi}(\boldsymbol{x}_r)\right]\underbrace{\langle \boldsymbol{j}\boldsymbol{j}^*\rangle}_{\boldsymbol{U}_s\widetilde{\boldsymbol{J}}_s(\omega)\boldsymbol{U}_s^T}\left[\boldsymbol{G}(\boldsymbol{x}_s,\boldsymbol{x}_r)+\boldsymbol{\Pi}(\boldsymbol{x}_r)\right]^*\boldsymbol{U}_{\|}\end{aligned}$$

Getting full data from polarization data (preprocessing)

- With shorthands $\widetilde{\boldsymbol{G}} = \boldsymbol{U}_{\|}^T \boldsymbol{G}(\boldsymbol{x}_s, \boldsymbol{x}_r)\boldsymbol{U}_s$ and $\widetilde{\boldsymbol{\Pi}} = \boldsymbol{U}_{\|}^T \boldsymbol{\Pi}(\boldsymbol{x}_r)\boldsymbol{U}_s$ we have

$$\begin{aligned}\psi(\boldsymbol{x}_r) &= [\widetilde{\boldsymbol{G}} + \widetilde{\boldsymbol{\Pi}}]\widetilde{\boldsymbol{J}}_s[\widetilde{\boldsymbol{G}} + \widetilde{\boldsymbol{\Pi}}]^* \\ &= \widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{G}}^* + \widetilde{\boldsymbol{\Pi}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{G}}^* + \widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{\Pi}}^* + \widetilde{\boldsymbol{\Pi}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{\Pi}}^*.\end{aligned}$$

- Asymptotic analysis of Kirchhoff imaging shows that only $\widetilde{\boldsymbol{\Pi}}$ is needed for imaging, i.e.

$$\mathcal{I}_{\mathrm{KM}}[\boldsymbol{\Pi}] \approx \mathcal{I}_{\mathrm{KM}}[\boldsymbol{U}_{\|}\widetilde{\boldsymbol{\Pi}}\boldsymbol{U}_s^T].$$

- Recovering $\widetilde{\boldsymbol{\Pi}}$ from coherency matrix data:

$$\widetilde{\boldsymbol{\Pi}} := (\psi(\boldsymbol{x}_r) - \widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{G}}^*)\widetilde{\boldsymbol{G}}^{-*}\widetilde{\boldsymbol{J}}_s^{-1} - \underbrace{(\widetilde{\boldsymbol{G}} + \widetilde{\boldsymbol{\Pi}})\boldsymbol{J}_s\widetilde{\boldsymbol{\Pi}}^*\widetilde{\boldsymbol{G}}^{-*}\boldsymbol{J}_s^{-1}}_{:=\boldsymbol{R}}.$$

- We show that Kirchhoff image of error term $\boldsymbol{R}$ is zero by a stationary phase argument similar to (Bardsley, GV 2016).
- Data preprocessing is thus:

$$\boxed{p(\psi) := \boldsymbol{U}_{\|}(\psi(\boldsymbol{x}_r) - \widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{G}}^*)\widetilde{\boldsymbol{G}}^{-*}\widetilde{\boldsymbol{J}}_s^{-1}\boldsymbol{U}_s^T.}$$

Kirchhoff imaging with polarization data

Theorem (Migration of preprocessed data $p(\psi)$)

Under a geometric assumption and as $k \to \infty$:

$$\mathcal{I}_{KM}[p(\psi)](\boldsymbol{y}) \approx \mathcal{I}_{KM}[\boldsymbol{\Pi}](\boldsymbol{y}).$$

Proof sketch. For error $\equiv \boldsymbol{U}_{\|}\widetilde{\boldsymbol{G}}\widetilde{\boldsymbol{J}}_s\widetilde{\boldsymbol{\Pi}}^*\widetilde{\boldsymbol{G}}^{-*}\boldsymbol{J}_s^{-1}\boldsymbol{U}_s^T$ we have

$$\mathcal{I}_{\mathrm{KM}}[\mathrm{error}](\boldsymbol{y}) = \int_{\mathcal{A}} d\boldsymbol{x}_{r,\|} \sum_{n=1}^{N} \boldsymbol{C}(\boldsymbol{x}_r, \boldsymbol{x}_s, \boldsymbol{y}_n, \boldsymbol{y}; k) \exp[\imath k \phi]$$

where $\boldsymbol{C}(\boldsymbol{x}_r, \boldsymbol{x}_s, \boldsymbol{y}_n, \boldsymbol{y}; k)$ is smooth, matrix valued. The phase ϕ is given by

$$\phi = 2|\boldsymbol{x}_r - \boldsymbol{x}_s| - |\boldsymbol{x}_r - \boldsymbol{y}_n| - |\boldsymbol{y}_n - \boldsymbol{x}_s| - |\boldsymbol{x}_r - \boldsymbol{y}| - |\boldsymbol{x}_s - \boldsymbol{y}|.$$

Geometric assumption $\implies \nabla_{\boldsymbol{x}_{r,\|}}\phi \neq 0 \implies \mathcal{I}_{\mathrm{KM}}[\mathrm{error}](\boldsymbol{y}) \to 0$
(stationary phase method).
$\rightsquigarrow$ similar argument for error term involving $\widetilde{\boldsymbol{\Pi}}$ and $\widetilde{\boldsymbol{\Pi}}^*$.

Stationary points of phase

To show error is small, enforce gradient wrt $\boldsymbol{x}_{r,\|}$ never vanishes i.e.

$$0 \neq \nabla_{\boldsymbol{x}_{r,\|}} \phi = 2\frac{\boldsymbol{x}_{r,\|} - \boldsymbol{x}_{s,\|}}{|\boldsymbol{x}_r - \boldsymbol{x}_s|} - \frac{\boldsymbol{x}_{r,\|} - \boldsymbol{y}_{n,\|}}{|\boldsymbol{x}_r - \boldsymbol{y}_n|} - \frac{\boldsymbol{x}_{r,\|} - \boldsymbol{y}_{\|}}{|\boldsymbol{x}_r - \boldsymbol{y}|}.$$

One way to guarantee this is:

Assumption (Geometric imaging condition)

If all scatterers are within region $\mathcal{W}$, we assume $\boldsymbol{x}_s$ satisfies

$$\frac{\boldsymbol{x}_{r,\|} - \boldsymbol{x}_{s,\|}}{|\boldsymbol{x}_r - \boldsymbol{x}_s|} \neq \frac{\boldsymbol{x}_{r,\|} - \boldsymbol{y}_{\|}}{|\boldsymbol{x}_r - \boldsymbol{y}|}$$

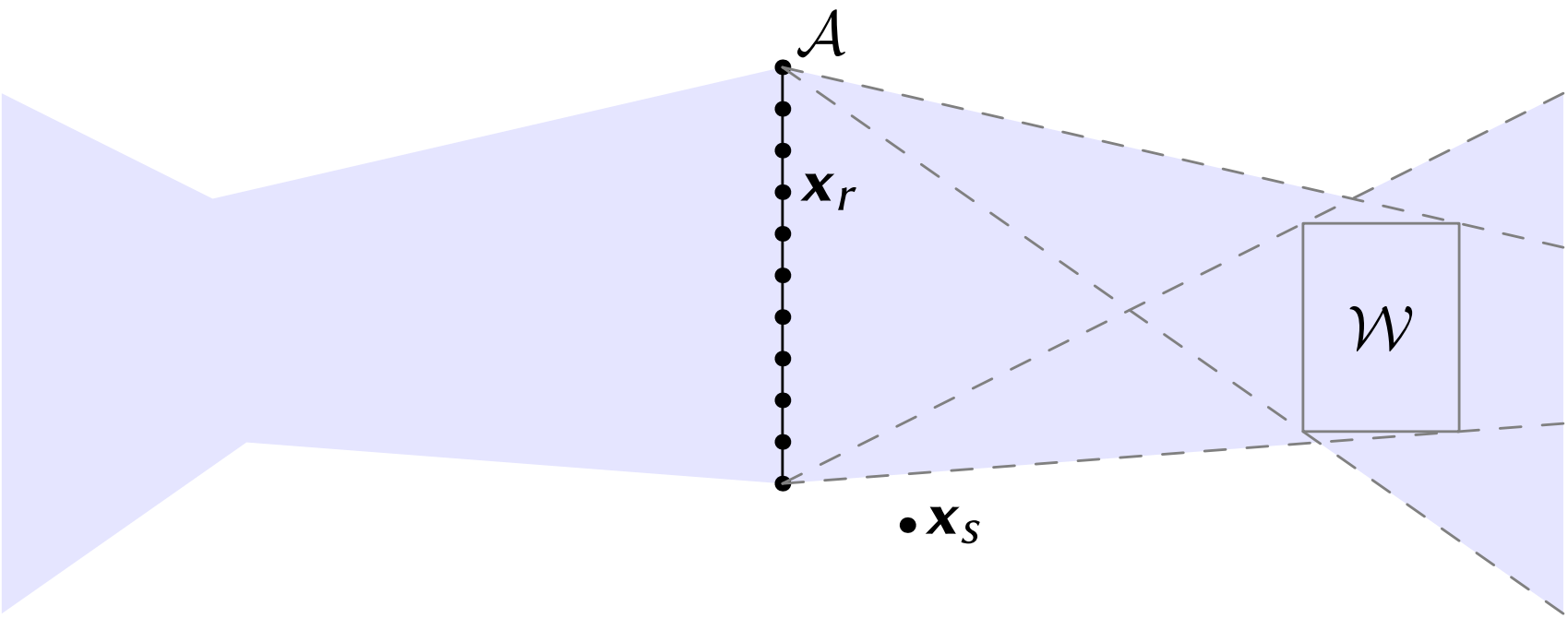
for all $\boldsymbol{x}_r \in \mathcal{A}$ and $\boldsymbol{y} \in \mathcal{W}$.

Source placement

Assumption (Geometric assumption)

For all $\mathbf{x}_r \in \mathcal{A}$, $\mathbf{y} \in \mathcal{W}$, *we assume* $\mathbf{x}_s$ *satisfies*

$$\frac{\mathbf{x}_{r,\|} - \mathbf{x}_{s,\|}}{|\mathbf{x}_r - \mathbf{x}_s|} \neq \frac{\mathbf{x}_{r,\|} - \mathbf{y}_\|}{|\mathbf{x}_r - \mathbf{y}|}$$



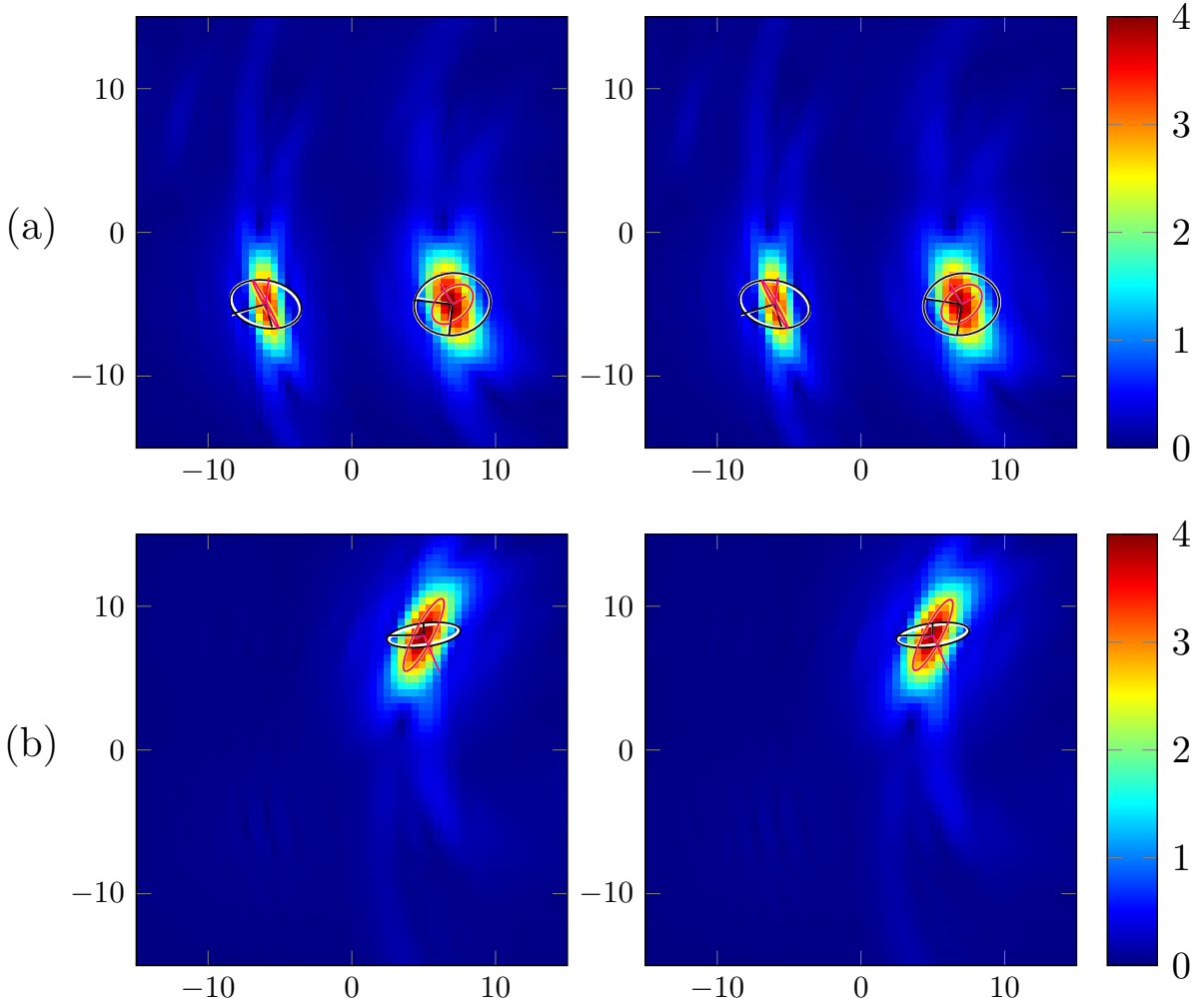
Upshot: source outside of blue region $\implies \mathcal{I}_{\mathrm{KM}}[p(\psi)](\mathbf{y}) \approx \mathcal{I}_{\mathrm{KM}}[\mathbf{\Pi}](\mathbf{y})$.

Numerical experiments

- $B = [1.2, 3.6]$ GHz ($\lambda_0 = 12.5$cm, $c = 3 \times 10^8 \text{ms}^{-1}$, $\Delta B = B/60$)
- $a = 20\lambda_0$ (sampled at 3 ppw)
- $L = 100\lambda_0$
- Source $\boldsymbol{x}_s = [50, 0, 100(1 - \sqrt{3}/4)]\lambda_0$, s.t. $|\boldsymbol{x}_s - \boldsymbol{y}_0| = 100\lambda_0$.
- Most experiments with three point scatterers given by

	$n = 1$	$n = 2$	$n = 3$
$\boldsymbol{y}_n$	$[-6, -5, 100]\lambda_0$	$[7, -5, 100]\lambda_0$	$[5, 8, 106]\lambda_0$
$\boldsymbol{\alpha}_n$	$\begin{bmatrix} 2+\imath & -\imath & 1 \\ -\imath & 1+2\imath & \imath \\ 1 & \imath & 1+\imath \end{bmatrix}$	$\begin{bmatrix} 2+2\imath & -1+\imath & \imath/2 \\ -1+\imath & 1+2\imath & 0 \\ \imath/2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2-2\imath & 1+\imath & 0 \\ 1+\imath & 1+2\imath & (1-\imath)/2 \\ 0 & (1-\imath)/2 & \imath \end{bmatrix}$

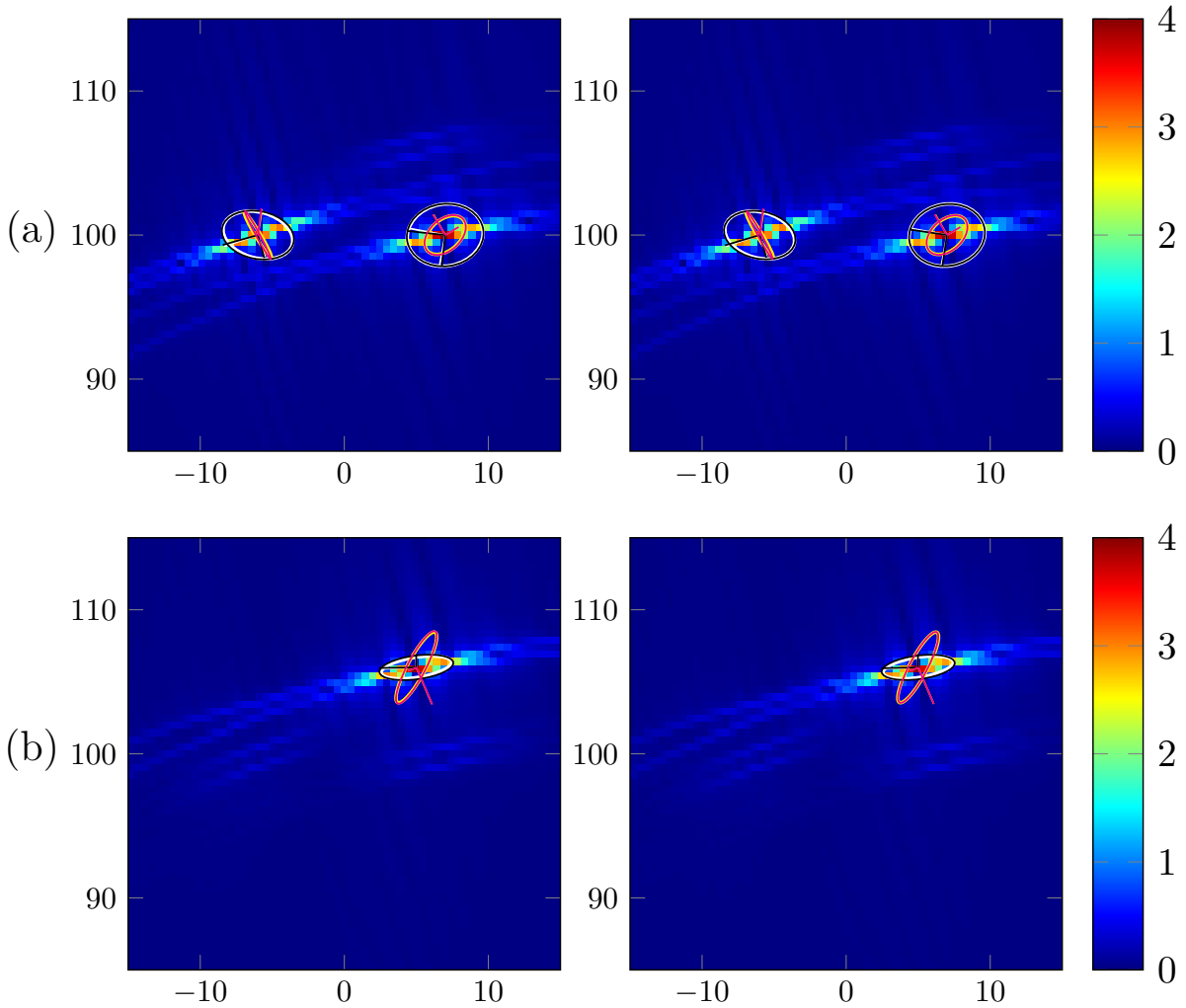
KM image in cross-range



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

(a) $z = 100\lambda_0$, (b) $z = 106\lambda_0$. left: true, right: reconstructed.

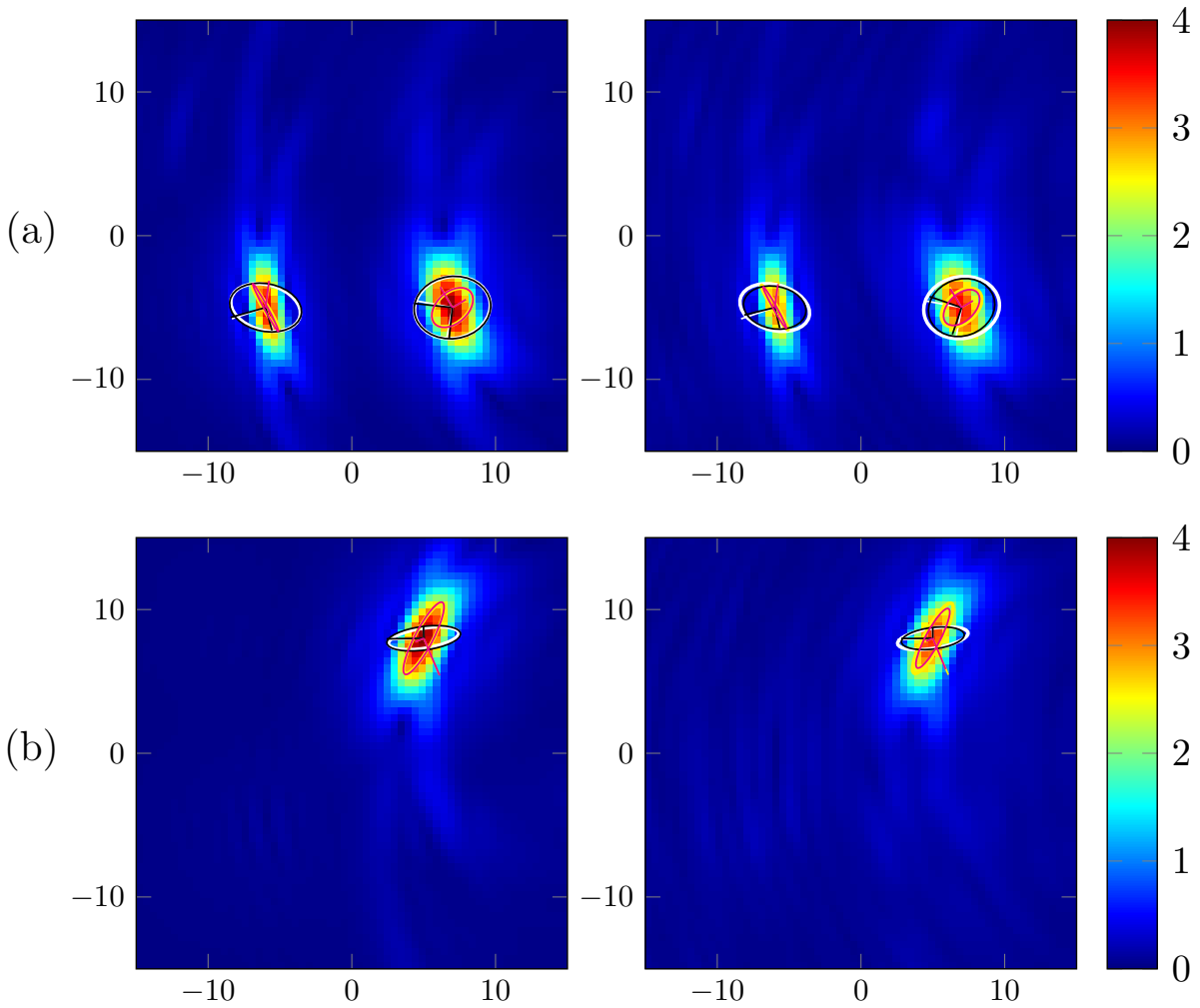
KM image in range



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

(a) $y = -5\lambda_0$, (b) $y = 8\lambda_0$. left: true, right: reconstructed.

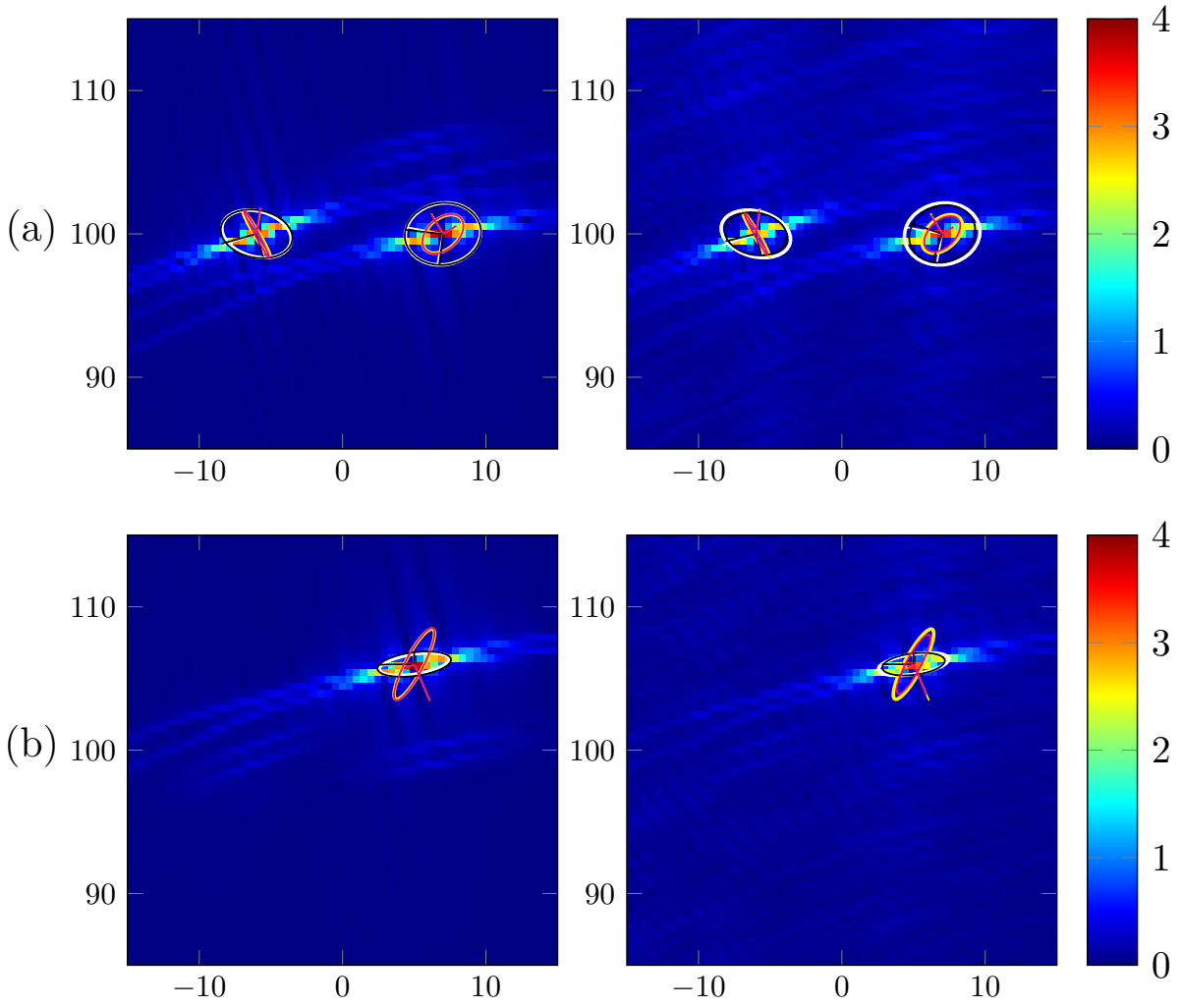
KM image in cross-range + stochastic acquisition



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

(a) $z = 100\lambda_0$, (b) $z = 106\lambda_0$. left: true, right: reconstructed.

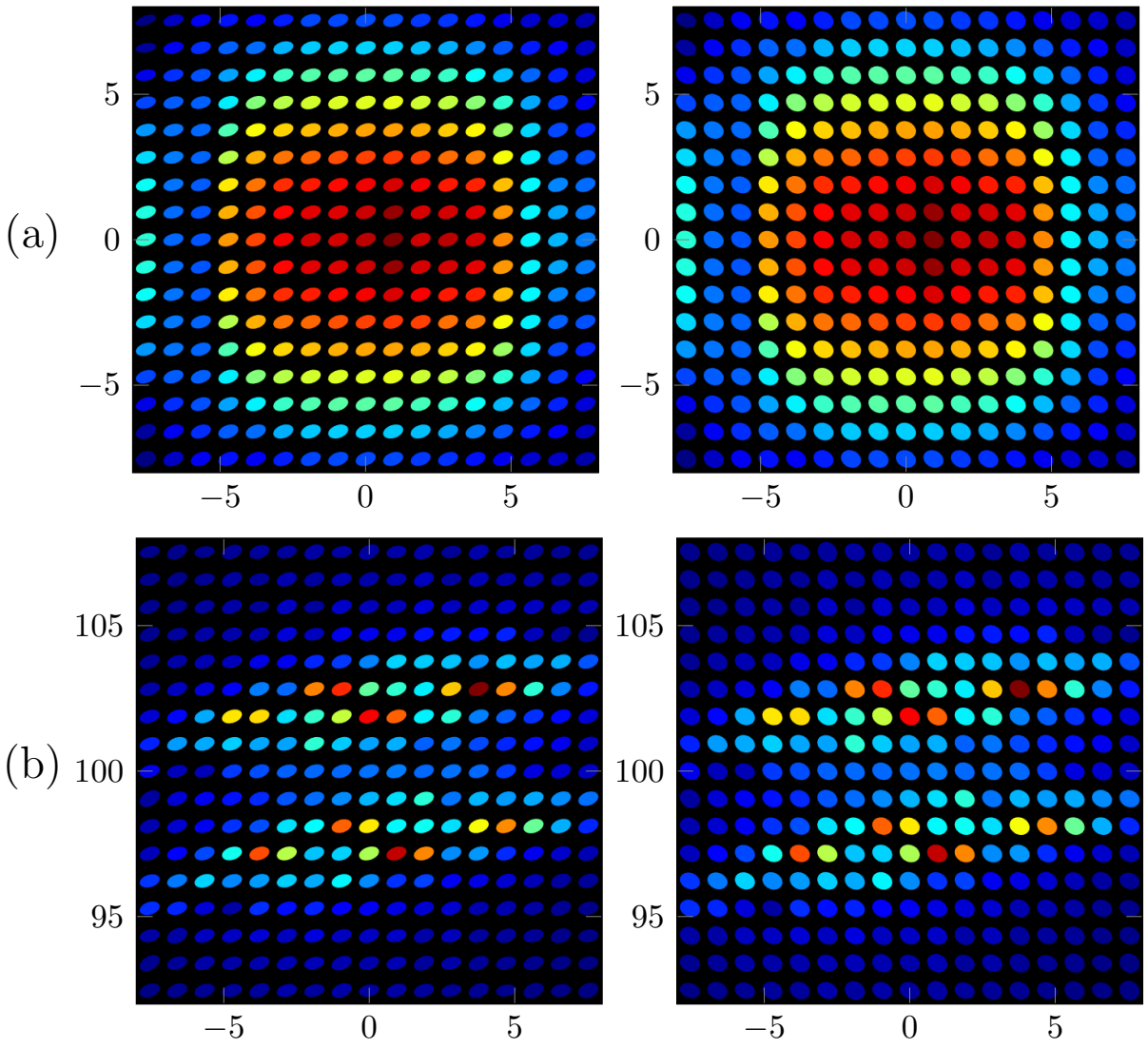
KM image in range + stochastic acquisition



	true	rec.
Re (α)	white	black
Im (α)	yellow	magenta

(a) $y = -5\lambda_0$, (b) $y = 8\lambda_0$. left: true, right: reconstructed.

KM image of $B([0,0,100]\lambda_0, 2.5\lambda_0)$ (stabilized)



(a): $z = 100\lambda_0$ (b): $y = 0$. left: real, right: imaginary. (fwd: 4ppw)

Summary

- Main idea is to pre-process polarization data and use existing imaging method.
- The source can be random with random dipole moment (white light) as long as polarization state for each wavelength is known.

Questions

- Can we incorporate multiple scattering with discrete dipole approximation? ($\equiv$ Foldy-Lax in acoustics)
- Can other phase retrieval methods be applied to polarization data?
- Can we use other reference beams?
- How to measure polarization at different wavelengths with just one sensor?

Thank you.

References

1. Bardsley and GV, "Kirchhoff imaging without phases". In: Inverse Problems 32.10 (2016), p. 105006.
   doi: 10.1088/0266-5611/32/10/105006.
   arXiv: 1601.02667 [math.NA].
2. Cassier and GV, "Imaging polarizable dipoles".
   In: SIAM J. Imaging Sci. 10.3 (2017), pp. 1381–1415.
   doi: 10.1137/17M112066X.
   arXiv: 1703.03544 [math.NA].
3. Bardsley, Cassier and GV, "Imaging with Stokes parameters".
   In preparation.