A non-linear stochastic inverse problem for faults in geophysics

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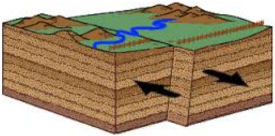
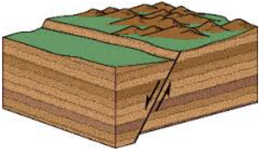
support: Simons foundation collaboration grant

Modeling faults and Slow Slip Events



strike slip



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or anything in between

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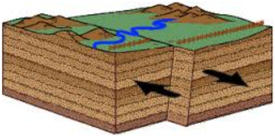

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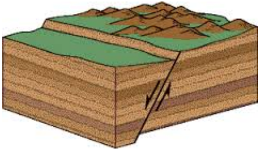

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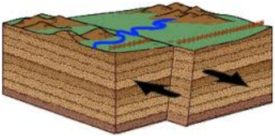

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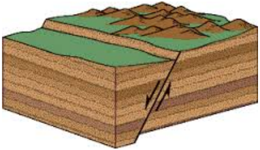

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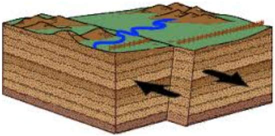

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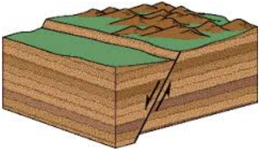

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- we model Slow Slip Events (SSEs) using half space linear elasticity
- displacement fields are **discontinuous** across active parts of faults
- Inverse problem: find the fault geometry, and the slip field on the fault (slip = discontinuity of displacements) from measurements of surface displacements

PDE model and integral formulation

Ω = half space $x_3 < 0$ minus a fault Γ, The displacement field $\mathbf{u}$ satisfies the forward fault problem

$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla \mathrm{div}\mathbf{u} = 0,$	in Ω	linear elasticity PDE
$T_{\mathbf{e}_3}(\mathbf{u}) = 0,$	on $x_3 = 0$	no force applied
$T_{\mathbf{n}}(\mathbf{u})$	continuous across Γ	continuity of forces
$[\mathbf{u}] = \mathbf{g}$	across Γ	given slip on Γ
([ ] denotes jumps)		
$\mathbf{u}$ decays at infinity		$\mathbf{u}$ has finite energy

we assume $\mathbf{g} \cdot \mathbf{n} = 0$ no opening or cross penetration

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integral formulation thanks to the adequate Green's tensor $\mathcal{H}$

$$\boxed{\mathbf{u} = \int_\Gamma \mathcal{H}\mathbf{g}}$$

Inverse problem

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- Given surface field $\mathbf{u}(x_1, x_2, 0)$, find the fault Γ and $\mathbf{g}$, the slip on Γ.

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- Given surface field $\mathbf{u}(x_1, x_2, 0)$, find the fault Γ and $\mathbf{g}$, the slip on Γ.
- but **mathematically** is that at all possible ???

Mathematical study: forward problem

Theorem (D.V. et al. *Inv. Prob.* 2017)

The forward fault problem (that is, the PDE) is uniquely solvable (in some adequate functional space $\mathcal{V}$).

The functional space $\mathcal{V}$

a Hardy type result (D.V. et al. *Inv. Prob.* 2017)

Theorem

Let Γ be a Lipschitz open surface which is strictly included in $\mathbb{R}^{3-}$. Let $\mathcal{V}$ be the space of vector fields $\mathbf{u}$ defined in $\mathbb{R}^{3-} \setminus \overline{\Gamma}$ such that $\nabla \mathbf{u}$ and $\dfrac{\mathbf{u}}{(1+|\mathbf{x}|^2)^{\frac{1}{2}}}$ are in $L^2(\mathbb{R}^{3-} \setminus \overline{\Gamma})$. Then the following four norms are equivalent on $\mathcal{V}$.

$$\|\mathbf{u}\|_1 = \Big(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla \mathbf{u}|^2\Big)^{\frac{1}{2}} + \Big(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} \frac{|\mathbf{u}|^2}{1+|\mathbf{x}|^2}\Big)^{\frac{1}{2}}, \|\mathbf{u}\|_2 = \Big(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\nabla \mathbf{u}|^2\Big)^{\frac{1}{2}}$$

$$\|\mathbf{u}\|_3 = \Big(\int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} |\epsilon(\mathbf{u})|^2\Big)^{\frac{1}{2}}, \|\mathbf{u}\|_4 = B(\mathbf{u},\mathbf{u})^{1/2}$$

where

$$B(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^{3-}\setminus\overline{\Gamma}} \lambda\, \mathrm{tr}(\nabla \mathbf{u})\mathrm{tr}(\nabla \mathbf{v}) + 2\mu\, \mathrm{tr}(\epsilon(\mathbf{u})\epsilon(\mathbf{v}))$$

Our uniqueness result for the inverse problem

Theorem (D.V. et al. *Inv. Prob.* 2017)

If $\mathbf{u}^1$ *and* $\mathbf{u}^2$ *solve the forward problem and are equal on a disk on the top surface* $\{x_3 = 0\}$ *then they correspond to the same fault and slip field.*

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- for related results see, Friedman Vogelius 1989 (2D conductivity in bounded domain), Beretta et al. 2008 (2D elasticity in bounded domain)

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- this result was proved for planar faults or faults included in two rectangles - more complicated geometries are possible
- for related results see, Friedman Vogelius 1989 (2D conductivity in bounded domain), Beretta et al. 2008 (2D elasticity in bounded domain)
- our problem: 3D, unbounded, and boundary conditions are given on the fault. "passive" inverse problem

Penalized deterministic reconstruction

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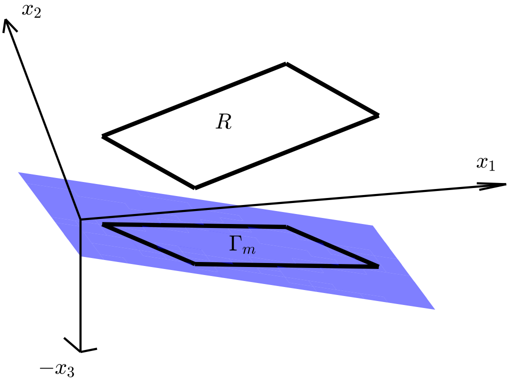
- Let R be a closed rectangle in the plane $x_3 = 0$.
- Let B be a set of $m = (a, b, d)$ such that the parallelogram

$$\Gamma_m = \{(x_1, x_2, ax_1 + bx_2 + d) : (x_1, x_2) \in R\}$$

is included in the half-space $x_3 < 0$



Define a regularized error functional

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- Let V be an open bounded set of $\{x_3 = 0\}$. Define the slip to surface displacements operator

$$\begin{aligned} A_m \quad : \quad & H_0^1(R) \to L^2(V) \\ & \mathbf{g} \to \int_R \mathcal{H}(\mathbf{x}, y_1, y_2, m)\mathbf{g}(y_1, y_2)\sigma dy_1 dy_2 \qquad \text{for } \mathbf{x} \in V \end{aligned}$$

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- for a fixed measurement $\tilde{\mathbf{u}}$ in $L^2(V)$ and a positive constant C, define the regularized error functional

$$F_{m,C}(\mathbf{g}) = \|\mathcal{C}^{-1/2}(A_m\mathbf{g} - \tilde{\mathbf{u}})\|^2_{L^2(V)} + C\|\mathbf{g}\|^2_{H_0^1(R)}$$

$\mathcal{C}$ is a uniformly positive definite matrix later interpreted as a covariance term

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- recall that B is the set of admissible geometry parameters m

Theorem

- Assume that $\tilde{\mathbf{u}} = A_{\tilde{m}}\tilde{\mathbf{h}}$ *for some* $\tilde{m}$ *in* B *and some* $\tilde{\mathbf{h}}$ *in* $H_0^1(R)$ *is the solution to the fault inverse problem.*
- Let C_n *be a sequence of positive numbers converging to zero.*
- Let m_n, $\mathbf{h}_{m_n,C_n}$ *be such that:*
$f(m_n, C_n) = \min_{m \in B} f(m, C_n)$, $f(m_n, C_n) = F_{m_n,C_n}(\mathbf{h}_{m_n,C_n})$.

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Then m_n *converges to* $\tilde{m}$, $\mathbf{h}_{m_n,C_n}$ *converges to* $\tilde{\mathbf{h}}$ *in* $H_0^1(R)$, *and* $A_{m_n}\mathbf{h}_{m_n,C_n}$ *converges to* $\tilde{\mathbf{u}}$ *in* $L^2(V)$.

Convergence rate

Proposition

The following convergence rates estimates hold

$$\|A_{m_n}\mathbf{h}_{m_n,C_n} - \tilde{\mathbf{u}}\| \leq C_n^{\frac{1}{2}}\|\tilde{\mathbf{h}}\|$$

$$\|\mathbf{h}_{m_n,C_n} - \tilde{\mathbf{h}}\| \leq \sqrt{2\|\mathbf{v}\|\|\tilde{\mathbf{h}}\|}(C_n^{\frac{1}{4}} + C(L,R)^{\frac{1}{2}}|m_n - \tilde{m})|^{\frac{1}{2}})$$

where we have assumed that $\tilde{\mathbf{h}}$ *is in the image of* $A^*_{\tilde{m}}$ *with* $\tilde{\mathbf{h}} = A^*_{\tilde{m}}\mathbf{v}$.

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- **work in progress:** once we establish stability results for the fault inverse problems we will be able to estimate the convergence rate of the geometry parameter $|m_n - \tilde{m}|$ in terms of C_n

Discrete error functional

Discrete error functional

- Given measurements $\tilde{\mathbf{u}}$ at the points P_j define the functional in V

$$F_{m,C}^{disc}(\mathbf{g}) = \sum_{j=1}^{N} C'(j,N)|\mathcal{C}^{-\frac{1}{2}}(A_m\mathbf{g} - \tilde{\mathbf{u}})(P_j)|^2 + C\int_R |\nabla \mathbf{g}|^2,$$

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- the terms $C'(j,N)$ are from a quadrature rule, while $\mathcal{C}$ is a covariance matrix and $C > 0$ is a constant
- to be minimized over $\mathcal{F}_p$, where $\mathcal{F}_p$ is an increasing sequence of finite-dimensional subspaces of $H^1_0(R)$ such that $\bigcup_{p=1}^{\infty}\mathcal{F}_p$ is dense.

Theorem: the solution to the discrete inverse problem converges to the true solution

Theorem (J. C. Sandiumenge, D. V. 2018)

Assume that $\tilde{\mathbf{u}}$ *was produced by the slip* $\tilde{\mathbf{h}}$ *for the geometry* $\tilde{m}$*. Set Let* $\overline{m}$ *in* B *be such that*

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Then $\overline{m}$ *tends to* $\tilde{m}$ *as* $C \to 0$, $N \to \infty$, $p \to \infty$ *provided* $N > cst(1/C)^{\beta}$, *where* β *is the order of the quadrature on* V.

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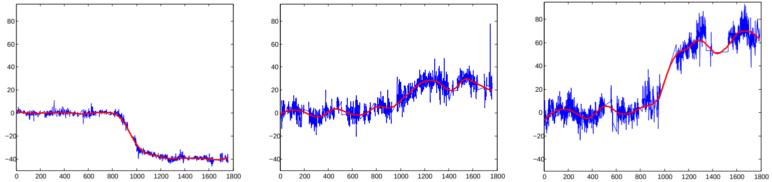
- Proof is involved since problem is non-linear in m
- this result shows that, in theory, finding deterministic solution $\overline{m}$ is possible
- but we would like to give a quantitative answer to the question: "if $\overline{m}'$ is close to $\overline{m}$, how likely is $\overline{m}'$ to be a solution"?

Stochastic modeling

Recorded surface displacements appear as



goal: account for uncertainties in data

Stochastic approach

Stochastic approach

- model:

$$(\tilde{\mathbf{u}}(P_1), \ldots, \tilde{\mathbf{u}}(P_N)) = (A_m\mathbf{g}(P_1), \ldots, A_m\mathbf{g}(P_N)) + \mathcal{E}$$

where m, $\mathbf{g}$, are random variable, $\mathcal{E}$ is Gaussian noise with density

$$\rho_{noise}(\mathbf{v}_1, \ldots, \mathbf{v}_N) \propto \exp(-\frac{1}{2}\sum_{j=1}^{N} C'(j, N)|\mathcal{C}^{-\frac{1}{2}}\mathbf{v}_j|^2)$$

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- Priors: we assume that $\mathbf{g}$ and m are independent with priors $\rho_{prior}(m) \propto 1_B(m), \quad \rho_{\mathcal{F}_p}(\mathbf{g}) \propto \exp(-\frac{1}{2} C \int_R |\nabla \mathbf{g}|^2)$

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- Baye's formula:

$$\rho(m, \mathbf{g} | \tilde{\mathbf{u}}_{meas}) \propto \rho(\tilde{\mathbf{u}}_{meas} | m, \mathbf{g}) \rho_{\mathcal{F}_p}(\mathbf{g}) \rho_{prior}(m)$$

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- to simplify notations Introducing the $3N$ by $3N$ diagonal matrix $\mathcal{D}$ such that

$$\mathcal{C}^{-\frac{1}{2}}(C'(1,N)^{\frac{1}{2}}\mathbf{u}(P_1), ..., C'(N,N)^{\frac{1}{2}}\mathbf{u}(P_N)) = \mathcal{D}(\mathbf{u}(P_1), ..., \mathbf{u}(P_N))$$

probability density of m knowing $\tilde{\mathbf{u}}_{meas}$

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- integrate $\rho(m, \mathbf{g}|\tilde{\mathbf{u}}_{meas})$ in $\mathbf{g}$: this can be done **exactly** to find

$$\rho(m|\tilde{\mathbf{u}}_{meas}) \propto \exp(-\frac{1}{2}F^{disc}_{m,C}(\mathbf{h}^{disc}_{m,C}))\frac{\rho_{prior}(m)}{\sqrt{\det((2\pi)^{-1}(A'_m\mathcal{D}^2A_m + CI_q))}}$$

where $F^{disc}_{m,C}$ is the same functional as earlier in this talk and $\mathbf{h}^{disc}_{m,C}$ is the point in $\mathcal{F}_p$ where it achieves its minimum

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  where $F^{disc}_{m,C}$ is the same functional as earlier in this talk and $\mathbf{h}^{disc}_{m,C}$ is the point in $\mathcal{F}_p$ where it achieves its minimum
- (loosely speaking) we proved that this posterior probability density must peak near $\tilde{m}$ (for small C, large N, small covariance)

Implementation and computation size

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- the discrete equivalent of minimizing $F_{m,C}^{disc}$ is to minimize

$$\|\mathcal{D}(Ag - u)\|^2 + C(\|Dg\|^2 + \|Eg\|^2)$$

- over g in $\mathbb{R}^q$
- D and E are in $\mathbb{R}^{q \times q}$ and discretize the derivatives in y_1 and y_2 (and are adjusted to be invertible)
- A is in $\mathbb{R}^{3N \times q}$ and depends on m
- $\mathcal{D}$ is in $\mathbb{R}^{3N \times 3N}$ and is diagonal

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  - $\mathcal{D}$ is in $\mathbb{R}^{3N\times 3N}$ and is diagonal
- **challenge**: due to the grid for m in $B \subset \mathbb{R}^3$ and iterations for finding C, this minimization problem had to be solved about 10^6 times

Find a way to choose the same C for all m

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- the matrix A depends on m_i, but we know that it is rectangular, has low rank, and a small number of rows
- in this problem A is a full matrix which is very expensive to compute due to the complexity of the Green's function $\mathcal{H}$

Parallel Algorithm

Algorithm for computing $C(i)$, $i \in I$

Compute and save $(D'D + E'E)^{-1}$

For each $i \in I$

    Compute A

    Compute the SVD of $\mathcal{D}A$

    if $\|\mathcal{D}(u - v)\| < \textbf{Err}$

        use a non-linear iterative solver to find $C(i)$

        % *at each iteration use Woodbury's formula for*

        % *computing* $(A'\mathcal{D}^2 A + C(D'D + E'E))^{-1} A'\mathcal{D}^2 u$

    else

        $C(i) = 0$

    end

end

Algorithm for computing the probability density $\rho(m|u)$

Let $M^2 = D'D + E'E$. The probability density of $\rho(m|u)$ is given by

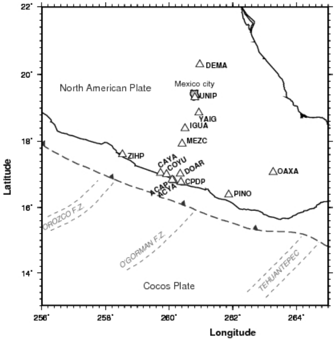
$$\mathcal{I}\exp(-\frac{1}{2}\|\mathcal{D}(Ag-u)\|^2 - \frac{1}{2}\mathbf{C}(\|Dg\|^2 + \|Eg\|^2))\frac{\rho_{prior}(m)}{\sqrt{\det((2\pi)^{-1}(A'\mathcal{D}^2A + \mathbf{C}M^2))}}, \qquad (1)$$

Algorithm for computing $\rho(m_i|u)/\mathcal{I}$,

Compute and save M^{-1} and $(D'D + E'E)^{-1}$
For each $i \in I$
    Compute A
    Compute the SVD of $\mathcal{D}A$
    Use the SV of $\mathcal{D}AM^{-1}$ to compute $\det((2\pi)^{-1}(A'\mathcal{D}^2A + \mathbf{C}I_q))$
    Solve $A'\mathcal{D}^2Ag + \mathbf{C}(D'D + E'E)g = A'\mathcal{D}^2u$
    Evaluate $\rho(m_i|u)/\mathcal{I}$ by using formula (1)
end

After applying the algorithm, there only remains to compute the normalizing constant $\mathcal{I}$

Application to a Slow Slip Event in Guerrero Mexico



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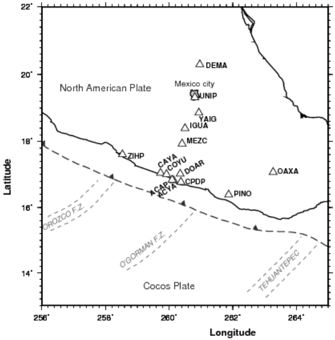

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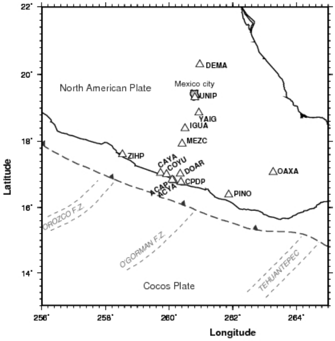
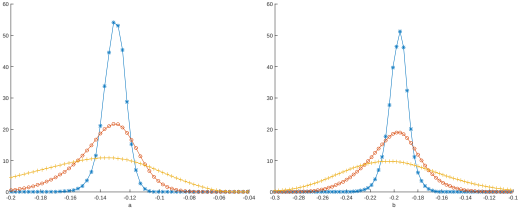


- the points P_j appear as GPS station on this map
- practice runs on simulated data **using these P_js**

Results: computed marginal probability densities



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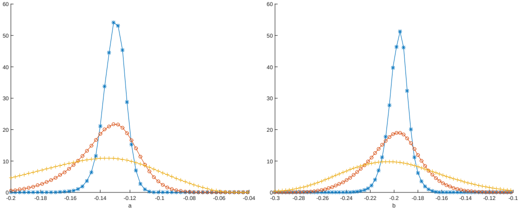

- Geometry parameters a and b ($m = (a, b, d)$, d shown on next slide) for the interface between plates in the Guerrero subduction zone.

  Shown: computed marginal distributions for the geometry parameters a, b. The blue star curve corresponds to the assumption that $\sigma_{hor} = .5, \sigma_{ver} = 1.5$, the red circle curve corresponds to the assumption that $\sigma_{hor} = 1, \sigma_{ver} = 3$, and the orange cross curve corresponds to the assumption that $\sigma_{hor} = 2, \sigma_{ver} = 6$.

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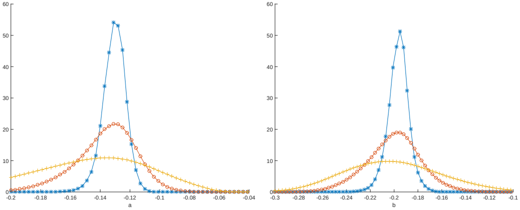

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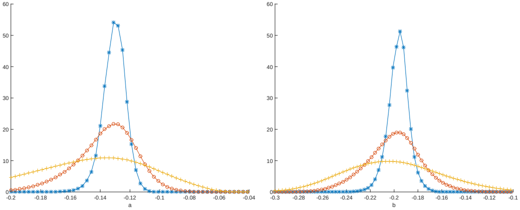

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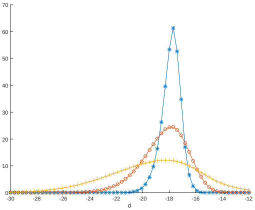
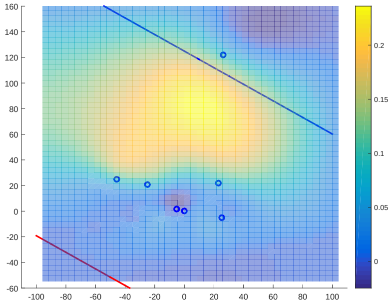
- best solution must be somewhere between blue and red ($\sigma_{hor}, \sigma_{ver}$ can only be estimated)
- Note that the probability densities for a, b, d CANNOT be computed separately since a, b, and d are not independent

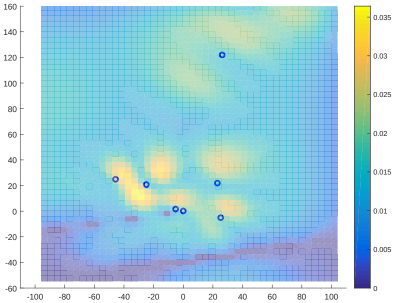


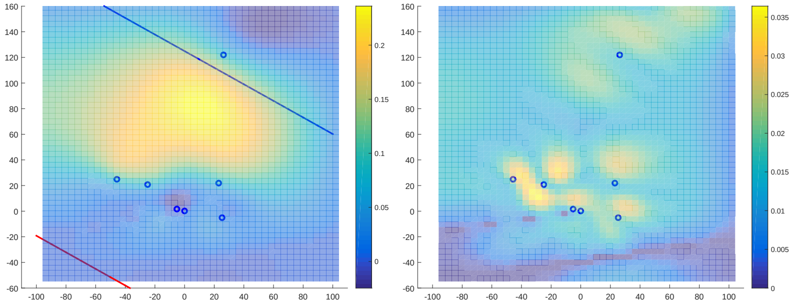
marginal distribution of d

Our results are consistent with other studies where geometries were found thanks to gravimetry or seismicity techniques (strike direction very close - location of active part also very close - more spread in dip angle)

Slip statistics for the average geometry profile







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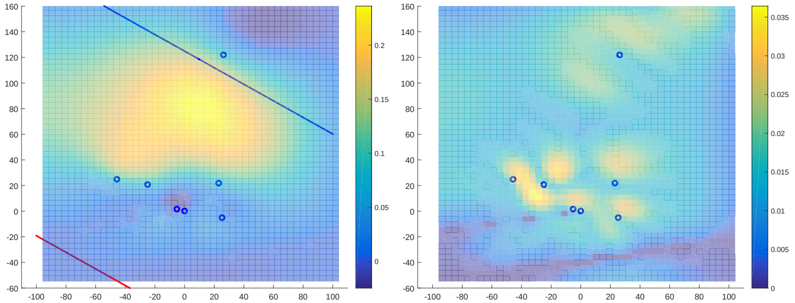

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- Classic (easier!) linear inverse problem, once we fix maximum likelihood geometry

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- Using stability results for the continuous problems derive convergence estimates of the recovered geometry as $C \to 0$ in the continuous and the discrete case
- New application in geophysics: SSE in the Cascadia region of Western North America: better imaging of related subduction zone. estimates of strain fields (size of computation and covariance of measurements will be different)