Inverse Source Problems in Elastodynamics

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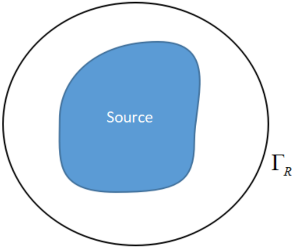
- Joint work with G. Bao, G. Hu and Y. Kian

Outline

- Motivation and problem formulation
- Scattering problems in elasticity
- Inverse source problems
- Numerical examples
- Future works



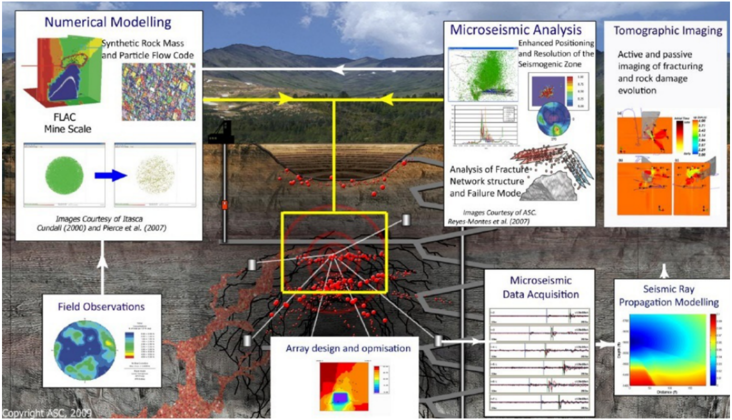
Source scattering problems are concerned with the relationship between radiating sources and wave fields.

- **Direct problem**: To determine the wave field from the given source and the differential equation governing the wave motion.
- **Inverse problem**: To determine the radiating source which produces the measured wave field.

Applications

- Non-destructive testing
- Teleseismic estimation
- Microseismic analysis
- Biomedical imaging

Application in microseismic



Elastic wave equations

- Equation of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} + F(x,t), \quad i = 1, \cdots, d.$$

- Constitutive equation

$$\sigma_{ij} = C_{ijkl} S_{kl} \quad \text{with} \quad S_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

The symmetry of σ_{ij} and S_{kl} yields

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}.$$

In isotropic medium

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are Lamé parameters satisfying $\mu > 0$ and $d\lambda + 2\mu > 0$.

In homogeneous isotropic medium

- Time-dependent Navier equation

$$\rho u_{tt} - \Delta^* u = F(x,t) \quad \text{in} \quad \mathbb{R}^d \times \mathbb{R}^+$$

- Time-harmonic Navier equation

$$\Delta^* u + \rho\omega^2 u = -F(x,\omega) \quad \text{in} \quad \mathbb{R}^d$$

Here, $\Delta^* = \mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}$ is the Lamé operator.

The Navier equation

$$\Delta^* u + \rho\omega^2 u = f \quad \text{in } \mathbb{R}^d.$$

The Helmholtz decomposition

$$u = \nabla\phi + \nabla \times \boldsymbol{\psi}, \quad \nabla \cdot \boldsymbol{\psi} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega},$$

where ϕ and $\boldsymbol{\psi}$ satisfy

$$\Delta\phi + k_p^2\phi = 0, \quad \nabla \times (\nabla \times \boldsymbol{\psi}) - k_s^2\boldsymbol{\psi} = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}.$$

Here

$$k_p = \omega\sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad k_s = \omega\sqrt{\frac{\rho}{\mu}}.$$

The Helmholtz decomposition

$$u = u_p + u_s \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega},$$

where

$$u_p = -\frac{1}{k_p^2}\nabla\nabla\cdot u, \quad u_s = \frac{1}{k_s^2}\nabla\times(\nabla\times u).$$

The Kupradze–Sommerfeld radiation condition:

$$\lim_{r\to\infty} r(\partial_r u_p - \mathrm{i}k_p u_p) = 0, \quad \lim_{r\to\infty} r(\partial_r u_s - \mathrm{i}k_s u_s) = 0, \quad r = |x|.$$

Consider the time-dependent problem

$$\rho U_{tt} - \Delta^* U = f(x)g(t) \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}^+,$$

with the initial conditions

$$U|_{t=0} = U_t|_{t=0} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

ISP. Let $f \in L^2(\mathbb{R}^3)^3$ be compactly supported in B_R for some $R > 0$ and $g \in \mathcal{C}(\mathbb{R})$ is supported in $[0, T_0]$. The ISP is to

- determine the spatial function f if g is known;
- determine temporal functions g if f is known.

- Vibration phenomena in seismology
- Teleseismic earthquake estimation

$$f(x) = exp(-a|x - x_0|^2)$$

$$g(t) = [1 - 2\pi^2\omega_0^2(t - t_0)^2]exp(-\pi^2\omega_0^2(t - t_0)^2)$$

- Biomedical imaging

$$\text{temporally localized:} \frac{d\delta(t)}{dt} f(x)$$

- Time-dependent acoustic: Bukhgaim & Klibanov ('81), Klibanov ('92), Yamamoto ('99), Khaĭdarov ('87), Isakov ('93,'98), Imanuvilov & Yamamoto ('01), Choulli & Yamamoto ('06), Kian, Sambou & Soccorsi ('16), Jiang, Liu & Yamamoto ('17), Fujishiro & Kian ('16),......
- Time-dependent elasticity: Ammari, Bretin, Garnier & Wahab ('09), Ammari, Bretin, Garnier, Kang, Lee & Wahab ('15),......
- Time-harmonic problems: Bleistein & Cohen ('77), Devaney & Sherman ('82), He & Romanov ('98), Hauer, Kühn & Potthast ('05), Albanese & Monk ('06), Badia & Nara ('11), Bao, Lin & Triki ('10), Bao, Lu, Rundell & Xu ('15), Zhang & Guo ('15), Cheng, Isakov & Lu ('16), Bao, Li, Lin & Triki ('15), Bao, Li & Zhao,......

By Helmholtz decomposition, the function $f \in (L^2(\mathbb{R}^3))^3$ supported in B_{R_0} admits a unique decomposition of the form

$$f(x) = \nabla f_p(x) + \nabla \times f_s(x), \quad \nabla \cdot f_s \equiv 0,$$

where $f_p \in H^1(B_{R_0})$, $f_s \in H_{\mathrm{curl}}(B_{R_0})$ also have compact support in B_{R_0}. Here,

$$H_{\mathrm{curl}}(B_{R_0}) := \{u : u \in (L^2(B_{R_0}))^3, \mathrm{curl}\ u \in (L^2(B_{R_0}))^3\}.$$

Lemma

Suppose that $S \in (L^2(\mathbb{R}^3))^3$ has a compact support in B_R for some $R > 0$, then the Helmholtz decomposition of S is unique.

By the completeness theorem, there exist vector-valued functions $U_p(x,t)$ and $U_s(x,t)$ such that $U(x,t)$ can be expressed as

$$U = U_p + U_s, \quad U_p = \nabla\, u_p, \quad U_s = \nabla \times\, u_s, \quad \nabla \cdot\, u_s = 0.$$

Moreover, the scalar function u_p and the vector function u_s satisfy the inhomogeneous wave equations

$$\frac{1}{c_\alpha^2}\,\partial_{tt}\, u_\alpha - \Delta u_\alpha = \frac{1}{\gamma_\alpha} f_\alpha(x) g(t) \quad \text{in} \quad \mathbb{R}^3 \times (0, +\infty), \qquad \alpha = p, s,$$

together with the initial conditions

$$u_\alpha\big|_{t=0} = \partial_t u_\alpha\big|_{t=0} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Note that

$$c_p := \sqrt{(\lambda + 2\mu)/\rho}, \quad c_s := \sqrt{\mu/\rho}, \quad \gamma_p := \lambda + 2\mu,\ \gamma_s := \mu,$$

and that $\lambda + 2\mu > 0$ since $\mu > 0$, $3\lambda + 2\mu > 0$. This implies that U_p and U_s propagate at different wave speeds, which will be referred as compressional waves (or simply P-waves) and shear waves (or simply S-waves), respectively.

It is well-known that the electrodynamic Green's tensor is given by

$$
\begin{aligned}
&G_{i,j}(x,t)\\
&= \frac{1}{4\pi\rho|x|^3}\left\{t^2\left(\frac{x_jx_k}{|x|^2}\delta(t-|x|/c_p)+(\delta_{jk}-\frac{x_jx_k}{|x|^2})\delta(t-|x|/c_s)\right)\right\}\\
&\quad+\frac{1}{4\pi\rho|x|^3}\left\{t\left(3\frac{x_jx_k}{|x|^2}-\delta_{jk}\right)(\Theta(t-|x|/c_p)-\Theta(t-|x|/c_s))\right\}.
\end{aligned}
$$

Using the above Green's tensor, the solution U to the inhomogeneous Lamé system can be represented as

$$
U(x,t)=\int_0^\infty\int_{\mathbb{R}^3}G(x-y,t-s)f(y)g(s)\,dxds,\quad x\in\mathbb{R}^3,\ t\in\mathbb{R}.
$$

Lemma

We have $U(x,t)\equiv 0$ for all $x\in B_R$ and $t>T_s:=T_0+(R+R_0)/c_s$.

Theorem

(i) The data set $\{U(x,t) : |x| = R, t \in (0, T_s)\}$ uniquely determines the spatial function f.
(ii) The data set of pure P- and S-waves, $\{U_\alpha(x,t) : |x| = R, t \in (0, T_\alpha)\}$, uniquely determines f_α $(\alpha = p, s)$.

This theorem remains valid if partial data $\{U_\alpha(x,t) : x \in \Gamma, t \in (0, T_s)\}$ $(\alpha = p, s)$ are available, where $\Gamma \subset \partial B_R$ is an arbitrary open subset. In fact, in the Fourier domain, the vanishing of $\hat{U}_\alpha(x,\omega)$ on Γ implies that $\hat{U}_\alpha(x,\omega) = 0$ on $|x| = R$ for each fixed $\omega \in \mathbb{R}^+$, due to the analyticity of the solution in a neighborhood of ∂B_R.

Determine the spatial function: frequency-domain approach

Introduce the functions

$$v_p(x,\omega) := d\, e^{-ik_p d\cdot x}, \quad v_s(x,\omega) := d^\perp\, e^{-ik_s d\cdot x}, \quad d \in \mathbb{S}^2.$$

Then we have

$$\begin{aligned}
&-\hat{g}(\omega)\int_{B_R} f(x)\cdot v_\alpha(x,\omega)\,dx \\
&= \int_{|x|=R} \Big[T_\nu \hat{U}(x,\omega)\cdot v_\alpha(x,\omega) - T_\nu v_\alpha(x,\omega)\cdot \hat{U}(x,\omega)\Big]\, ds,
\end{aligned}$$

and

$$\int_{B_R} f(x)\cdot v_p(x,\omega)\,dx = ik_s(2\pi)^{\frac{3}{2}}\ \hat{f}_p(k_p d),$$

$$\int_{B_R} f(x)\cdot v_s(x,\omega)\,dx = ik_s(2\pi)^{\frac{3}{2}}\ \hat{f}_s(k_s d)\cdot(d\times d^\perp).$$

We apply Duhalme's principle to U by setting

$$U(x,t) = \int_0^t V(t-s,x)g(s)\,ds, \quad x \in \mathbb{R}^3, t > 0.$$

The function V then fulfills the homogeneous Lamé equation with non-zero initial conditions

$$\begin{aligned}
&\partial_{tt} V(x,t) = -c_p^2\, \nabla \times \nabla \times V(x,t) + c_s^2\, \nabla(\nabla \cdot V(x,t)), \\
&V(x,0) = 0, \quad \partial_t V(x,0) = f(x).
\end{aligned}$$

Then we decouple V into the sum of the compressional part V_p and shear part V_s their initial condition are related to $\nabla f_p(x)$ and $\nabla \times f_s(x)$, respectively.

The compactly supported function f is called a non-radiating source at the frequency $\omega \in \mathbb{R}^+$ to the Lamé system if there exists a $P \in \mathbb{C}^3$ such that the unique radiating solution to the inhomogeneous Lamé system

$$\Delta^* u(x) + \omega^2 \rho u(x) = f(x)\, P, \quad j = 1, 2, 3,$$

vanishes identically in $\mathbb{R}^3 \backslash \overline{\mathrm{supp}(f)}$.

Theorem

Suppose that f is not a non-radiating source for all $\omega \in \mathbb{R}^+$. Then the temporal function $g \in \mathcal{C}_0([0, T_0])^3$ can be uniquely determined by the partial boundary measurement data $\{U(x,t) : x \in \Gamma, t \in (0, T_s)\}$ where $\Gamma \subset \partial B_R$ is an arbitrary subboundary with positive Lebesgue measure.

Consider the inverse problem of determining g from observations of the solution at one fixed point $x_0 \in \mathrm{supp}(f)$.

Theorem

Let $x_0 \in B_R$, $p > 5/2$ and consider $M, \delta > 0$ such that

$$\mathcal{A}_{x_0,p,\delta,M} := \{h \in H^p(\mathbb{R}^3) : \ \|h\|_{H^p(\mathbb{R}^3)} \leq M, \ |h(x_0)| \geq \delta\} \neq \emptyset.$$

Then, for $f \in \mathcal{A}_{x_0,p,\delta,M}$, it holds that

$$\|g\|_{L^2(0,T)^3} \leq C \, \|\partial_{tt} U(x_0, \cdot)\|_{L^2(0,T)^3}$$

where C depends on λ, μ, ρ, p, x_0, M, R, δ and T. In particular, this estimate implies that the data $\{U(x_0, t) : t \in (0, T)\}$ determines uniquely the temporal function g.

Remark

From the mathematical point of view, the recovery of source terms of the form $g(t)f(x)$ is the best we can expect.

- There is no hope to recover general source terms of the form $F(x, t)$.
- There is even an obstruction for the recovery of source terms of the form $g_1(t)f_1(x) + g_2(t)f_2(x)$.

Remark

Let $\chi = (\chi_1, \chi_2, \chi_3)$ with $\chi_j \in \mathcal{C}_0^\infty(B_{R_0} \times (0, T_0))$, $j = 1, 2, 3$. Now fix

$$F(x,t) := \rho \partial_{tt} \chi - \mathcal{L}_{\lambda,\mu} \chi$$

and consider the problem

$$\begin{cases} \rho \partial_{tt} U(x,t) = \mathcal{L}_{\lambda,\mu} U(x,t) + F(x,t), & (x,t) \in \mathbb{R}^3 \times (0, +\infty), \\ U(x,0) = \partial_t U(x,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1}$$

Clearly $U = \chi$ is the unique solution of (1). Assuming that $\chi \neq 0$, from the uniqueness of solutions of (1) one can check that $F \neq 0$. However since $\mathrm{supp}(\chi) \subset B_{R_0} \times (0, T_0)$, we have

$$U(x,t) = 0, \quad |x| = R, \ t \in (0, +\infty)$$

and $\mathrm{supp}(F) \subset B_{R_0} \times (0, T_0)$, but $F \neq 0$. This proves that the data $\{U(x,t) : \ |x| = R, \ t > 0\}$ do not allow to recover general sources $F(x,t)$ satisfying $\mathrm{supp}(F) \subset B_{R_0} \times (0, T_0)$.

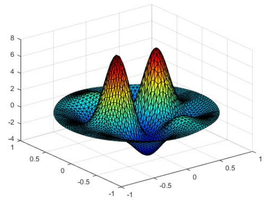
- Reconstruction of f

$$\hat{U}(x,\omega)/\hat{g}(\omega) = \int_{B_R} \hat{G}(x-y)\, f(y)dy, \quad |x| = R, \quad \hat{g}(\omega) \neq 0,$$

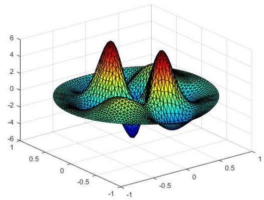
- Reconstruction of g

$$I_1(\omega_i) = [W(x_{0,i},\omega_i)]^{-1}\hat{U}(x_{0,i},\omega_i)$$

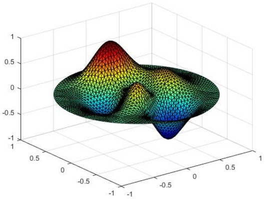
$$I_2(\omega_i) := \frac{1}{M}\sum_{j=1}^{M}[W(x_{j,i},\omega_i)]^{-1}\hat{U}(x_{j,i},\omega_i), \quad i = 1, 2, \cdots, K,$$



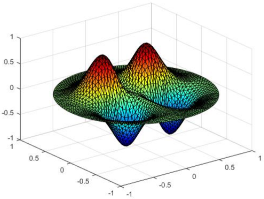
(a) f_1



(b) f_2

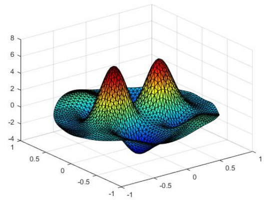


(c) f_p

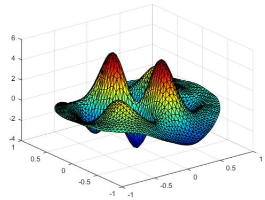


(d) f_s

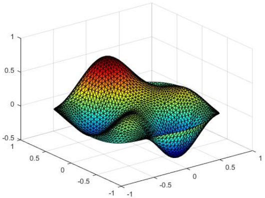
Figure: The exact spatial source function $f = (f_1, f_2)$ and its compressional component f_p and shear component f_s.



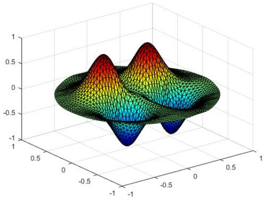
(b) Reconstructed f_1



(c) Reconstructed f_2



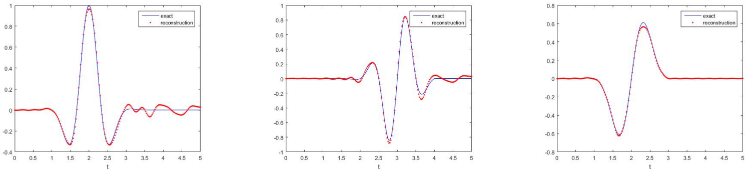
(e) Reconstructed f_p



(f) Reconstructed f_s

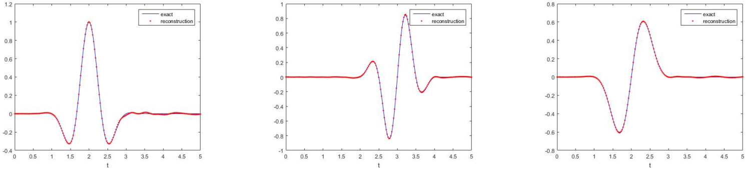
Figure: Reconstructions of f from Fourier-transformed time-domain data.

Numerical examples



(a) g_1 (a) g_2 (a) g_3

Figure: Reconstruction of temporal functions from l_1 with 30% noise.



(a) g_1 (a) g_2 (a) g_3

Figure: Reconstruction of temporal functions from l_2 with 30% noise.

Ongoing and future works

- Inverse source problems for elastic scattering in porous medium
- Inverse source problems for fluid-solid and fluid-bone interaction problems
- Inverse medium problems in elasticity

Thank You !